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Priestley duality for distributive semilattices

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Abstract

In this paper, we extend Priestley duality for bounded distributive lattices to all bounded distributive semilattices. We show that we cannot take the prime spectrum as Priestley dual but have to turn to a suitable weakening of the concept of prime ideal.

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Introduction

Stone duality for Boolean algebras in 1936 ([10]) and Pontryagin duality for abelian groups ([8]) are major achievements in algebra that opened the door for numerous and fruitful developments : Stone-like dualities exist now in abundance, as nicely shown in Davey's paper ([4] and [2], see also [3] and [7]). One of the most interesting works in this aera is Priestley's duality for distributive lattices in 1970 ([9]) where a full comprehension of natural dualities was perhaps for the first time made possible.

Another feature of Priestley duality is that it gives a simple alternative to Stone duality for distributive lattices. Now Stone duality very easily extends to distributive semilattices and finds in this larger context its most natural setting, as shown by Grätzer in 1971 ([6]). In this paper we show that rather curiously such an extension is not possible stricto sensu as far as Priestley duality is concerned. We give also an alternative solution in terms of a spectrum that is larger than the prime spectrum and examine some classical dualisations.

1 The duality

Definitions 1.1. 1) By a *distributive semilattice*, we mean a bounded $\vee$-semilattice $S = (S, \vee, 0, 1)$[1] such that

$$(Dist) c \leq a \vee b \Rightarrow \exists a' \leq a, \exists b' \leq b \quad \text{with} \quad c = a' \vee b'.$$

If the formula $(Dist)$ is not an equation in the language $\vee, 0, 1$, it becomes so in an extended language $\vee, 0, 1, \downarrow$, where $\downarrow$ is the (multivalued) unary operation such that $a \downarrow = \{b \mid b \leq a\}$. Hence the class of distributive semilattices is closed under cartesian products, subobjects and homomorphic images if homomorphisms were understood as mappings respecting $\vee, 0, 1$ and $\downarrow$ (that is, $f(a \downarrow) = (f(a)) \downarrow$, like p-morphisms of Kripke structures, see [1]). Unfortunately, we shall have to consider other maps as morphisms to get a duality. We do this now, following Grätzer ([6]).

2) If S and S' are distributive semilattices and f is a map $S \to S'$, then f is said to be a *morphism* if

$$(Mor) P' \text{ prime ideal of } S' \Rightarrow f^{-1}(P') \text{ prime ideal of } S,$$

where we recall that an ideal of a semilattice is *prime* if its complement is non-empty and lower directed.

It is not difficult to prove that f is a morphism if and only if it respects $\vee, 0, 1$ and satisfies the following "equation" :

$$(f(a \downarrow \cap b \downarrow)) \downarrow = f(a) \downarrow \cap f(b) \downarrow .$$

We denote the resulting category by $\mathcal{S}$ and recall that Grätzer has shown that $\mathcal{S}$ is dually equivalent to the category of Stone spaces (compact sober spaces with a basis of compact open sets) and *strongly continuous* maps (continuous maps such that the inverse image of a compact open set is compact).

At the object level, the ("Stone") dual of a distributive semilattice S, its *Stone space*, is $(X_1(S), \tau_1)$ where $X_1(S)$ is the prime spectrum of S

[1] As usual, we use the same symbol to denote a structure and its universe. When necessary, the intended type will be clear from the categorical context.

(the set of prime ideals of S), endowed with the topology τ_1 generated by $\{\tau_1(a) \mid a \in S\}$ with

$$\tau_1(a) = \{P \in X_1(S) \mid P \not\ni a\}.$$

3) Following Priestley's ideas, we define the *prime Priestley space* of S as $\mathcal{X}_1(S) = (X_1(S), \tau, \leq)$ where the order is the inclusion relation and τ is the topology generated by the $\tau_1(a)$ $(a \in S)$ and their complements.

   We shall see that this space is usually not compact, that its decreasing clopen subsets are not necessarily of the form $\tau_1(a)$, for some $a \in S$, even not an intersection of a finite number of them. More dramatically, we shall show that $\mathcal{X}_1(S)$ does not characterize S. In the quest for a suitable dual, it is natural to turn to some compactification of $\mathcal{X}_1(S)$. At the algebraic level, this amounts to consider the free distributive lattice over a distributive semilattice, and we first do this.

Construction 1.2. Let S be a distributive semilattice. Denote by $Fin(S)$ the set of all finite subsets of S and for $E = \{e_1, \ldots, e_n\} \in Fin(S)$, let $I_E = \bigcap\{e_i \downarrow \mid i = 1, \ldots, n\}$.

Define an equivalence θ on $Fin(S)$ by $E\theta F$ if and only if $I_E = I_F$. Order the quotient $F(S) = Fin(S)/\theta$ by $E^\theta \leq F^\theta$ if $I_E \subseteq I_F$. To characterize $F(S)$ we need to introduce weak morphisms of semilattices : we say that a map $f : S \to S'$ between distributive semilattices is a *weak morphism* if it respects $0, 1, \vee$ and $\wedge$ when the latter is defined (this is indeed weaker than Definition 1.1, but of course, stronger than the usual semilattice homomorphisms, that we do not consider in this paper). We denote $\mathcal{S}^W$ the resulting category.

Theorem 1.3. *With the notations of 1.2, $F(S)$ is the free distributive lattice over S within $\mathcal{S}^W$.*

Proof. It is clear that $F(S)$ is an ordered set in which $E^\theta \wedge F^\theta$ is given by $(E \cup F)^\theta$. We use the distributivity of S to compute $\vee$. If $E, F \in Fin(S)$, let $E \vee F = \{e \vee f \mid e \in E \text{ and } f \in F\}$. Then $E^\theta \vee F^\theta = (E \vee F)^\theta$. To prove this, first note that $E^\theta \leq (E \vee F)^\theta$ and $F^\theta \leq (E \vee F)^\theta$. Also, if $E^\theta, F^\theta \leq G^\theta$, then $I_{E\vee F} \subseteq I_G$. Otherwise there exists $x \in I_{E\vee F}$ and $g \in G$ such that $x \not\leq g$. Let P be a prime ideal such that $P \ni g$ and $P \not\ni x$. If both E and F meet P, let $e \in P \cap E$ and $f \in P \cap F$. Then, $x \leq e \vee f \in P$, which is impossible. Hence either $P \cap E = \emptyset$ or $P \cap F = \emptyset$, say $P \cap E = \emptyset$. Since P is prime, there is some z with $z \in I_E$ and $z \notin P$. Hence $z \in I_E \subseteq I_G$, whence $z \leq g$, which prevents $g \in P$, another contradiction.

We have proved that $F(S)$ is a lattice. In fact the mapping $E^\theta \mapsto I_E$ embeds $F(S)$ into the ideal lattice of S. In particular, $F(S)$ is distributive.

Let $\alpha : S \to F(S)$ be the natural map : $s \mapsto \{s\}^\theta$. Clearly, α is a weak morphism (but generally not a morphism, see Remark 1.4 3)). Suppose now h is a weak morphism from S into a distributive lattice L. The only possible extension of h to the whole of $F(S)$ is

$$\tilde{h} : E^\theta \mapsto \bigwedge\{h(e) \mid e \in E\}.$$

We show that this definition works, that is, $\tilde{h}$ is a lattice homomorphism $F(S) \to L$ such that $\tilde{h} \circ \alpha = h$. The only non-trivial point is to prove that $\tilde{h}$ is well-defined, that is, $I_E = I_F$ implies $\bigwedge\{h(e) \mid e \in E\} = \bigwedge\{h(f) \mid f \in F\}$. It clearly suffices to show that if $I_E \subseteq f\downarrow$, then $\bigwedge\{h(e) \mid e \in E\} \leq h(f)$. From $I_E \subseteq f\downarrow$ follows $E^\theta \leq \{f\}^\theta$, whence $(E \vee \{f\})^\theta = \{f\}^\theta$ and $\bigwedge\{e \vee f \mid e \in E\}$ exists in S and equals f. Since h is a weak morphism, we have $h(f) = \bigwedge\{h(e \vee f) \mid e \in E\} = \bigwedge\{h(e) \vee h(f) \mid e \in E\} = (\bigwedge\{h(e) \mid e \in E\}) \vee h(f)$ by distributivity, and therefore $\bigwedge\{h(e) \mid e \in E\} \leq h(f)$ as required. □

This result calls for some remarks.

Remarks 1.4. 1) If $\mathcal{D}$ is the category of bounded distributive lattices, then Theorem 1.3 is the statement that $\mathcal{D}$ is a reflective subcategory of $\mathcal{S}^W$ via the functor F.

2) In the proof of the previous theorem, one sees that $F(S)$ is a sublattice of the ideal lattice of S. More is seen in fact : $F(S)$ is the sublattice of the ideal lattice of S generated by (the principal ideals of) S and any element of $F(S)$ is a meet of finitely many elements of S : $E^\theta = \bigwedge\{\{e\}^\theta \mid e \in E\}$ holds for any $E \in Fin(S)$.

3) The corresponding result in $\mathcal{S}$ is a negative one. In fact, the free distributive lattice over S exists in $\mathcal{S}$ if and only if S is itself a lattice. To prove this, suppose on the contrary that L is a free distributive lattice over S in $\mathcal{S}$ and let $\tau : S \to L$ be the canonical morphism. By Theorem 1.3, there is a lattice homomorphism $\tilde{\tau} : F(S) \to L$ such that $\tilde{\tau} \circ \alpha = \tau$. Since $\tau(S)$ generates L, $\tilde{\tau}$ is onto. Let us show that $\tilde{\tau}$ is moreover one-to-one. This amounts to show that if $\bigwedge\{\tau(e) \mid e \in E\} \leq \tau(f)$,

then $I_E \subseteq f\downarrow$. If this is not the case, there is some $t \in I_E$ with $t \not\leq f$. But this implies $\tau(t) \not\leq \tau(f)$, whence $\bigwedge\{\tau(e) \mid e \in E\} \not\leq \tau(f)$, a contradiction. This shows that $F(S) \cong L$ and α is a morphism, and not only a weak morphism. Suppose now S is not a lattice. Then for some finite $E \subseteq S$, $\bigwedge E$ does not exist and I_E is not a principal ideal. Hence, in $F(S)$ there exists a prime ideal P' that contains all $\alpha(i), i \in I_E$, but no $\alpha(e), e \in E$. By the property 1.1 2) defining morphisms, $P = \alpha^{-1}(P')$ is prime in S. This is not possible since $\alpha^{-1}(P') \cap E = \emptyset$ but $\alpha^{-1}(P') \supseteq I_E$.

4) By the above remark, the inverse image of a prime ideal of $F(S)$ is not necessarily prime in S. It is not difficult to characterize those ideals of S that arise as inverse images of prime ideals in $F(S)$.

Lemma 1.5. *If P' is a prime ideal of $F(S)$, then $P = \alpha^{-1}(P')$ $(= \{s \in S \mid \{s\}^\theta \in P'\})$ is an ideal of S such that*

$$(WP) \forall e_1, \ldots, e_n \notin P,\ \forall f \in P, \exists t \leq e_1, \ldots, e_n \quad \textit{with} \quad t \not\leq f.$$

Proof. We have $\alpha(e_1), \ldots, \alpha(e_n) \notin P'$, whence $\bigwedge_{i=1}^n \alpha(e_i) \notin P'$ and since $\alpha(f) \in P'$, $\bigwedge_{i=1}^n \alpha(e_i) \not\leq \alpha(f)$. If E denotes $\{e_1, \ldots, e_n\}$, this can be read $E^\theta \not\leq \{f\}^\theta$, that is $I_E \not\subseteq f\downarrow$. Hence there is $t \in I_E$, that is $t \leq e_1, \ldots, e_n$ such that $t \not\leq f$, as required. □

Definition 1.6. An ideal satisfying axiom (WP) of the previous Lemma is called *weakly prime*. It is interesting to compare axiom (WP) with the following axiom (P) that defines prime ideals among ideals of a distributive semilattice :

$$(P)\ \forall e_1, \ldots, e_n \notin P \exists t \leq e_1, \ldots, e_n,\ \forall f \in P, t \not\leq f.$$

Also, this definition is easily seen to be equivalent with that of a weak prime element in the ideal lattice $\mathcal{I}d(S)$ of S as given in [5], p. 81. As a consequence, the set $X(S)$ of all weak prime ideals of S is closed in the Lawson topology (see [5], p. 246). As such, it is a compact ordered space. Since $\mathcal{I}d(S)$ is morever algebraic, we get a Priestley space. Let us make things more precise and give direct proofs in the following lines.

If S a distributive semilattice, we denote by $X(S) = (X(S), \tau, \leq)$ the ordered space of weak prime ideals of S, where $\leq$ is inclusion and τ is the topology generated by the sets $r(a)$ $(a \in S)$ and their complements, where

$$r(a) = \{P \in X(S) \mid P \not\ni a\}.$$

This topology is the topology induced by the Lawson topology on $\mathcal{I}d(S)$, as a consequence of Exercise 1.12, p. 147, in [5].

Theorem 1.7. *The ordered space $X(S)$ is a Priestley space whose dual lattice is nothing else than $F(S)$.*

Proof. It is clear that $X(S)$ is t.o.d.; to have a direct proof of its compactness we argue classically, as in the lattice case, using the following adaptation of the prime ideal theorem : if I is a proper ideal of S and F is an increasing subset of S such that $I_E \not\subseteq i\downarrow$ whenever $E \in Fin(F)$ and $i \in I$, then there exists a weak prime ideal P of S such that $P \supseteq I$ and $P \cap F = \emptyset$.

Let us now prove that $X(S)$ is isomorphic to the Priestley dual $X(F(S))$ of $F(S)$, by the dual map of α, that is by $\alpha^* : P' \mapsto \alpha^{-1}(P')$. Lemma 1.5 ensures that α^* is a map $X(F(S)) \to X(S)$. Now we show that α^* is onto. Let P be a weakly prime ideal of S. Then P is the inverse image through α of a unique prime ideal of $F(S)$, namely $Q = \alpha(P)\downarrow (= \bigcup\{\alpha(p)\downarrow \mid p \in P\})$. The only thing to prove is that Q is prime in $F(S)$. If not, then $E^\theta, F^\theta \notin Q$ but $E^\theta \wedge F^\theta \in Q$ for some $E, F \in Fin(S)$. This means $E \cap P = \emptyset$, $F \cap P = \emptyset$ and $I_{E\cup F} \subseteq p\downarrow$ for some $p \in P$, which contradicts axiom (WP) of 1.5. The other verifications are routine, as well as the continuity of α^*, which follows from the formula

$$\alpha^*(r(a)) = r(\alpha(a)).$$

□

From this theorem, it follows that $X(S)$ does not characterize S, and we have announced that $X_1(S)$ does not characterize S either (this will be proved in Corollary 2.2). The pair of them does, however, as we proceed to show.

Definitions 1.8. 1) Let $\mathcal{X} = (X, X_1)$ be a structure such that $X = (X, \tau, \leq)$ is an ordered space and $X_1 \subseteq X$. A decreasing open set O is said to be of *type 1* if $O \cap X_1$ is cofinal in O (for a clopen set O in a compact ordered space, it amounts to asking that any maximal element of O be in X_1).

We denote by $\mathcal{D}(\mathcal{X})$ or $\mathcal{D}(X)$ the set of all clopen decreasing subsets of X and by $\mathcal{D}_1(\mathcal{X})$ the set of all $O \in \mathcal{D}(\mathcal{X})$ of type 1. An element $x \in X$ is said to be of *type 1* if $\{O \in \mathcal{D}_1(\mathcal{X}) \mid O \ni x\}$ is a basis of decreasing clopen neighborhoods at x.

The structure $\mathcal{X}$ is of *type 1* if it satisfies

$$x \in X_1 \Leftrightarrow x \text{ is of type 1.}$$

We also recall that a subset X_1 of an ordered topological space X is *order-generating* if for all $x \in X, x \downarrow \cap X_1$ is non-empty and has x as infimum ([5], p. 70). It is equivalent to ask that i) if X has a greatest element 1, then $1 \in X_1$ and ii) if $y \not\leq x$, then $y \not\leq x_1$ for some $x_1 \in X_1$ such that $x \leq x_1$. Moreover, if X_1 is order-generating, any element of X is dominated by some element of X_1 and the whole space X is a clopen decreasing subset of type 1.

We say that $\mathcal{X} = (X, X_1)$ is a *Priestley structure* if

i) X is a Priestley space,

ii) X_1 is dense and order-generating in X,

iii) $\mathcal{X}$ is of type 1.

2) We turn the class of all Priestley structures into a category $\mathcal{P}$ by defining a *morphism* between Priestley structures $\mathcal{X} = (X, X_1)$ and $\mathcal{X}' = (X', X_1')$ to be a continuous order-preserving map $h : X \to X'$ such that

i) $x \in X_1$ implies $h(x) \in X_1'$, and

ii) $h^{-1}(O')$ is of type 1 whenever $O' \in \mathcal{D}_1(\mathcal{X})$.

3) If $S \in \mathcal{S}$, its *dual structure* is $\mathcal{X}(S) = (X(S), X_1(S))$ and if $\mathcal{X} \in \mathcal{P}$, its *dual semilattice* is $\mathcal{D}_1(\mathcal{X})$ ($\mathcal{D}_1(X)$ is clearly a semilattice when ordered by inclusion).

Lemma 1.9. *If $\mathcal{X} = (X, X_1)$ is a Priestley structure, it satisfies the following improvement of the t.o.d. axiom :*

(t.o.d$_1$) if $x \not\leq y$, there exists $O \in \mathcal{D}_1(\mathcal{X})$ such that $O \ni y$ and $O \not\ni x$.

Proof. This follows directly from order-generation and the fact that the space is t.o.d. and of type 1. □

Lemma 1.10. *If $S \in \mathcal{S}$ and $O \in \mathcal{D}(X(S))$, then $O = r(e_1) \cap \ldots \cap r(e_n)$ for some $e_1, \ldots, e_n \in S$. Moreover, O is of type 1 if and only if $O = r(a)$ for some $a \in S$.*

Proof. In the proof of Theorem 1.7, we defined an isomorphism α^* from $X(F(S))$ onto $X(S)$, whence it follows that $\alpha^{*-1}(O) \in \mathcal{D}(X(F(S)))$ and by Priestley duality $\alpha^{*-1}(O) = r(E^\theta)$ for some $E^\theta \in F(S)$. This gives the first assertion with $E = \{e_1, \ldots, e_n\}$.

To prove the second assertion, suppose first $Q \in r(a)$ where $a \in S$. Then there is a prime P such that $Q \subseteq P$ and $P \not\ni a$. This shows that $r(a)$ is of type 1. Conversely, suppose $O = r(e_1) \cap \ldots \cap r(e_n)$ is not of the form $r(a)$ for any $a \in S$. Then $E = \{e_1, \ldots, e_n\}$ has no infimum. If Q is any weakly prime ideal such that $I_E \subseteq Q$ and $Q \cap E = \emptyset$, then $Q \in O$ but Q is not contained in any prime ideal P such that $P \in O$. Hence O is not of type 1. □

Lemma 1.11. *If $S \in \mathcal{S}$, its dual structure $\mathcal{X}(S)$ is a Priestley structure. And if $\mathcal{X}$ is a Priestley structure, its dual semilattice $\mathcal{D}_1(\mathcal{X})$ is distributive.*

Proof. 1) From Theorem 1.7 we know that $X(S)$ is a Priestley space. Assertion ii) in Definition 1.8 follows from 1.6 and [5], p. 71 and p. 247. It remains to prove iii).

If $P \in X_1(S)$, that is, P is prime, and $O \in \mathcal{D}(X(S))$ is such that $P \in O$, then by the preceding Lemma, there are $e_1, \ldots, e_n$ such that $O = r(e_1) \cap \ldots \cap r(e_n)$. Since $P \in O$, we have $e_1, \ldots, e_n \notin P$ and since P is prime, there is $a \notin P$ with $a \leq e_1, \ldots, e_n$. Hence $P \in U \subseteq O$ for some $U = r(a)$ of type 1, as required.

Conversely, we must show that if P is of type 1, then P is prime. Let $e, f \notin P$. Then $P \in r(e) \cap r(f) \in \mathcal{D}(X(S))$. Since P is of type 1, there is by the preceding Lemma some a such that $P \in r(a) \subseteq r(e) \cap r(f)$. This means that $a \notin P$ and $a \leq e, f$, whence P is prime.

2) We now prove that $\mathcal{D}_1(\mathcal{X}) \in \mathcal{S}$ if $\mathcal{X} \in \mathcal{P}$. First $\mathcal{D}_1(\mathcal{X})$ is bounded, with greatest element X by axiom ii) of 1.8. To prove distributivity, let $O, U, V \in \mathcal{D}_1(\mathcal{X})$ be such that $O \subseteq U \cup V$. Denote by $\max(O)$ the set of maximal elements of O. Then for each $x \in \max(O)$, either $x \in U$ and there is O_x of type 1 such that $x \in O_x \subseteq O \cap U$, or $x \in V$ and there is O_x of type 1 such that $x \in O_x \subseteq O \cap V$. The various O_x $(x \in \max(O))$ form a covering of O, from which we can extract a finite subcovering, say $O_1, \ldots, O_n$ with $O_i \subseteq O \cap U$ for $i \leq n$ $(\leq m)$ and $O_i \subseteq O \cap V$ otherwise. Let $U_1 = \bigcup\{O_i \mid i \leq n\}$ and $V_1 = \bigcup\{O_i \mid n < i \leq m\}$. Then $O = U_1 \cup V_1$ with $U_1 \subseteq U$ and $V_1 \subseteq V$.

□

Theorem 1.12. *The categories $\mathcal{S}$ and $\mathcal{P}$ are dually equivalent. The dual of the free distributive lattice over S is the underlying Priestley space of the dual of S.*

Proof. We first extend the object mappings $\mathcal{X} : \mathcal{S} \to \mathcal{P}$ and $\mathcal{D}_1 : \mathcal{P} \to \mathcal{S}$ to functors in the usual way : $\mathcal{X}(f) = f^{-1}$ and $\mathcal{D}_1(h) = h^{-1}$. We first have to prove that $\mathcal{X}(f)$ and $\mathcal{D}_1(h)$ are morphisms.

It is even not clear that $\mathcal{X}(f)$ is a mapping, that is, $f^{-1}(Q)$ is a weakly prime ideal whenever Q is. If $e_1, \ldots, e_n \notin f^{-1}(Q)$ and $a \in f^{-1}(Q)$, there is a $t \leq f(e_1), \ldots, f(e_n)$ with $t \not\leq f(a)$. Since f is a morphism, there is $e \leq e_1, \ldots, e_n$ with $t \leq f(e)$, and necessarily $e \not\leq a$ otherwise $t \leq f(e) \leq f(a)$. The other verifications about $\mathcal{X}(f)$ are trivial or come from the equation

$$(\mathcal{X}(f))^{-1}(r(a)) = r(f(a)).$$

We show that $\mathcal{D}_1(h)$ satisfies the equation (Mor) in 1.1 2). Suppose $O \subseteq h^{-1}(U) \cap h^{-1}(V)$ $(O, U, V \in \mathcal{D}_1(\mathcal{X}))$. Then $h(O) \subseteq U \cap V$. For each $x' \in \max(h(O))$, there is $x \in \max(O)$ with $x' = h(x)$. Since $x \in \max(O)$, x is of type 1, and so is x'. Hence there is $W_{x'}$ of type 1 with $x' \in W_{x'} \subseteq U \cap V$. By the compactness of $h(O)$, $h(O)$ is covered by finitely many $W_{x'}$: $h(O) \subseteq W = W_{x'_1} \cup \ldots \cup W_{x'_n} \subseteq U \cap V$ and $O \subseteq h^{-1}(W)$ with $W \subseteq U \cap V$.

Using Lemma 1.10, it is easy to see that the map $r_S : S \to \mathcal{D}_1(\mathcal{X}(S))$: $a \mapsto r(a)$ is an isomorphism and the composition $\mathcal{D}_1\mathcal{X}$ is naturally equivalent to the identity.

Before considering the composition $\mathcal{X}\mathcal{D}_1$, we establish the second assertion : if $\mathcal{X} = (X, X_1)$, we have to prove that $\mathcal{X}F(\mathcal{D}_1(\mathcal{X})) \cong (X, X_1)$ or equivalently, by Priestley duality, that $F(\mathcal{D}_1(\mathcal{X}) \cong \mathcal{D}(X)$. We use the notations of 1.2 and 1.3 : the correspondence $(O_1, \ldots, O_n\}^\theta \mapsto O_1 \cap \ldots \cap O_n$ is clearly an order embedding from $F(\mathcal{D}_1(\mathcal{X}))$ into $\mathcal{D}(X)$ and it remains to show that it is onto. Let $U \in \mathcal{D}(X)$. For $y \in U$ and $x \notin U$, we have $x \not\leq y$ and by Lemma 1.9, there is $O \in \mathcal{D}_1(\mathcal{X})$ with $O \ni y$ and $O \not\ni x$. A standard double compactness argument shows that U is of the form $U = O_1 \cap \ldots \cap O_n$ for some $O_i \in \mathcal{D}_1(\mathcal{X})$.

Now an isomorphism $\xi_{\mathcal{X}} : \mathcal{X} \mapsto \mathcal{X}\mathcal{D}_1(\mathcal{X})$ is given by the composition of the Priestley space isomorphism $X \to \mathcal{X}\mathcal{D}(X) : x \mapsto P_x = \{O \in \mathcal{D}(X) \mid O \not\ni x\}$, the isomorphism $\mathcal{X}\mathcal{D}(X) \to \mathcal{X}F(\mathcal{D}_1(\mathcal{X}))$ given above and the isomorphism $\alpha^* : \mathcal{X}F(\mathcal{D}_1(\mathcal{X})) \to \mathcal{X}\mathcal{D}_1(\mathcal{X})$ given in Theorem 1.7. This gives the map $\xi_{\mathcal{X}} : x \mapsto \{O \in \mathcal{D}_1(\mathcal{X}) \mid O \not\ni x\}$ at the Priestley space level and it remains to show that

$$x \in X_1 \Leftrightarrow \xi_{\mathcal{X}}(x) \in X_1(\mathcal{D}_1(\mathcal{X})).$$

Suppose first $x \in X_1$. Then the weakly prime ideal $\{O \in \mathcal{D}_1(\mathcal{X}) \mid O \not\ni x\}$ is prime because if $O_1, \ldots, O_n \ni x$, then there is some O of type 1 such that $x \in O \subseteq O_1 \cap \ldots \cap O_n$. Suppose now $x \notin X_1$. Since $\mathcal{X}$ is of type 1, there is some $U \in \mathcal{D}(\mathcal{X})$ with $x \in U$ but for no $O \in \mathcal{D}_1(\mathcal{X})$ one has $x \in O \subseteq U$. By a previous argument, $U = O_1 \cap \ldots \cap O_n$ where $O_i \in \mathcal{D}_1(\mathcal{X})$ for all i. Hence $\{O \in \mathcal{D}_1(\mathcal{X}) \mid O \not\ni x\}$ is not prime. □

2 Examples and dualisations

We first want to prove a fact that was announced in the introduction : the Priestley topology (and order) on the prime spectrum of a distributive semilattice does not characterize it. We need a general method for constructing Priestley structures with a unique accumulation point.

Lemma 2.1. *Let $P = P_1 \dot\cup \infty$ be an infinite ordered set, topologized as the one-point Alexandroff compactification of P_1, considered as a discrete topological space. Then (P, P_1) is a Priestley structure if and only if*

a) $\infty \leq x$ for any x with $x\downarrow$ infinite,

b) $x \leq \infty$ for any x with $x\uparrow$ infinite,

c) ∞ is not a greatest element of P,

d) $\infty\uparrow$ is finite, and

e) if $x \not\leq \infty$, then $x \not\leq y$ for some $y > \infty$.

Proof. It is not difficult to show that P is a Priestley space if and only if conditions a) and b) hold.

Suppose now (P, P_1) is a Priestley structure. Since P is of type 1, any maximal element of P is in P_1 and in particular c) holds. Also, since $\infty \notin P_1$, ∞ is not of type 1 by axiom iii) and some decreasing clopen $V \ni \infty$ is not of type 1 and therefore admits a maximal element not of type 1. This means that ∞ is maximal in V, whence $\infty\uparrow \setminus\{\infty\} \subseteq -V$. Since ∞ is the only point of accumulation of P, $-V$ is finite and d) holds. Finally if $x \not\leq \infty$, then $x \not\leq \bigwedge(\infty \uparrow \setminus\{\infty\})$ by axiom ii) and e) follows.

Conversely, suppose (P, P_1) satisfies all conditions of the Lemma. Then P is a Priestley space (by a) and b)) in which P_1 is dense. To show that P_1 is order generating, it suffices to verify $\infty = \bigwedge(\infty \uparrow \setminus\{\infty\})$, which is equivalent to c). Let $x \in P_1$. If $\infty \not\leq x$, then $x \uparrow$ is finite by a), and $\{x \downarrow\}$ is a basis of clopen decreasing neighborhoods of type 1 at x. If $\infty < x$, then any clopen decreasing neighborhoods of x is of type 1, whence x is of type 1 in this case also. Finally, let us choose for each $x > \infty$ a clopen decreasing V_x containing ∞ but not x. By c) and d), $V = \bigcap\{V_x \mid x > \infty\}$ is a clopen decreasing neighborhood of ∞ that contains ∞ as a maximal element, and that prevents ∞ from being of type 1. □

As a corollary, we now give two non isomorphic distributive semilattices S and T such that $X_1(S) \cong X_1(T)$ (but of course $X(S) \not\cong X(T)$).

Corollary 2.2. *Let P_1 be the cardinal sum of an infinite antichain and a diabolo $D = \{a, b, c, d\}$ with $a \leq c$, $a \leq d$, $b \leq c$, $b \leq d$. Then there exist at least two non isomorphic Priestley spaces P and P' such that (P, P_1) and (P', P_1) are Priestley structures.*

Proof. We just refer to Lemma 2.1 and define $P = P_1 \dot\cup \{\infty\}$, $P' = P_1 \dot\cup \{\infty'\}$ with $a, b \leq \infty \leq c, d$ and $\infty' \leq a, b, c, d$.

Note that the dual semilattices are not difficult to describe. Let L be the distributive lattice that is the dual of the diabolo $D = \{a, b, c, d\}$ as above, so that $L = \{0, 2, 3, 4, 5, 6, 1\}$ with $0 = \emptyset$, $1 = D$, $2 = a \downarrow$, $3 = b \downarrow$, $4 = a \downarrow \cup b \downarrow$, $5 = c \downarrow$ and $6 = d \downarrow$; and let B be the Boolean algebra of finite and cofinite subsets of $\mathbb{N}$. Then the distributive semilattice S that is dual to (P, P_1) can be realized as

$$S = \{(i, x) \in L \times B \mid x \text{ finite } \Leftrightarrow i = 0, 2, 3 \text{ or } 4\}$$

while $S' = \mathcal{D}_1(P', P_1)$ is

$$S' = \{(i, x) \in L \times B \mid x \text{ finite } \Leftrightarrow i = 0\}.$$

□

Our next task is a description of coproducts in $\mathcal{S}$, using duality. It is an appealing fact that Stone duality is required at some intermediate steps of our argument.

In the following lemma, we use notations $r_1(a)$ of 1.12 and $r(a)$ of 1.6.

Lemma 2.3. *If $S \in \mathcal{S}$ and $a, a_1, \ldots, a_n, b \in S$, then*

1) $\overline{r_1(a)} = r(a)$ *and* $\overline{-r_1(a)} = -r(a)$,

2) $\overline{r_1(a) \cap \ldots \cap r_1(a_n) \setminus r_1(b)} = r(a_1) \cap \ldots \cap r(a_n) \setminus r(b)$.

Proof. 1) Inclusions $\subseteq$ are obvious. Thus $\overline{r_1(a)}$ and $\overline{-r_1(a)}$ form a partition of $X(S)$ and 1) is proved.

2) We have $\subseteq$ by 1). Let $Q \in r_1(a) \cap \ldots \cap r(a_n) \setminus r(b)$ and let O be a clopen neighborhood of Q. By Lemma 1.10, we may suppose that $O = r(c_1) \cap \ldots \cap r(c_m) \setminus r(d)$. Hence $Q \not\ni a_1, \ldots, a_n, \ldots, c_1, \ldots, c_m$ and $Q \ni b \vee d$. Because Q is weakly prime, there is $e \leq a_1, \ldots, c_m$ with $e \not\leq b \vee d$. By Stone separation theorem there is a prime ideal P with $P \not\ni e$ and $P \ni b \vee d$. This shows that O meets $r_1(a_1) \cap \ldots \cap r_1(a_n) \setminus r_1(b)$, as required.

□

Let for $i \in I$, S_i be a distributive semilattice, let $\mathcal{X}_i = ((X_i, \tau_i, \leq), X_{1i})$ be its Priestley dual, (X_{1i}, τ_{1i}) be its Stone dual. Denote by X the cartesian product of all Priestley spaces $(X_i, \tau_i, \leq)$, by Y the cartesian product of all Stone spaces (X_{1i}, τ_{1i}) and by S the semilattice of compact open subsets of Y (Stone dual of S). Then we have the following result.

Lemma 2.4. *With the above notations, the mapping*

$$\xi : X \to X(S) : x \mapsto \{O \in S \mid \overline{O} \not\ni x\}$$

is an isomorphism (of Priestley spaces).

Proof. a) Let us first show that ξ is a map, that is, $\xi(x)$ is weakly prime. Suppose $O_1, \ldots, O_n \notin \xi(x)$ and $U \in \xi(x)$: we have to find $W \in S$ with $W \subseteq O_1 \cap \ldots \cap O_n$ and $W \not\subseteq U$.

Remember that each $V \in S$ is a finite union of products $\prod_i V_i$ where each V_i is compact open in (X_{1i}, τ_{1i}), and $V_i = X_{1i}$ for all but finitely many i. Let V be one of the O_j ($j = 1, \ldots, n$). Then $x \in \overline{V}$ hence x

is in the closure of some product $\prod_i V_i$: let us call it $\prod_i V_i^{(j)}$. If on the other side V is U, then $x \notin \overline{U} = \overline{\bigcup_k \prod_i U_i^{(k)}}$ and for each k there is some i_k with $x_{i_k} \notin \overline{U_{i_k}^{(k)}}$. For each i, let $Z_i = V_i^{(1)} \cap \ldots \cap V_i^{(n)} \setminus \bigcup \{U_{i_k}^{(k)} \mid i_k = i\}$. By Lemma 2.3, $x_i \in \overline{Z_i}$ for each i, whence $Z_i \neq \emptyset$. Choose a point w_i in each Z_i. Because (X_{1i}, τ_{1i}) is a Stone space, there is some compact open W_i in X_{1i} with $w_i \in W_i \subseteq V_i^{(1)} \cap \ldots \cap V_i^{(n)}$. Let $W = \prod_i W_i$. By construction, W fulfills all requirements : $W \in S, W \subseteq O_1 \cap \ldots \cap O_n$ and $W \not\subseteq U$ (since $w = (w_i) \in W$ and $w \notin U$).

b) It is clear that ξ is one-to-one : if $x \neq y$ in X, there is some clopen decreasing subset U of X that separates x from y. By Lemma 2.3, we may assume that $U = \overline{O}$ for some $O \in S$. Hence $\xi(x) \neq \xi(y)$.

Let us show that ξ is onto. If $\mathcal{P}$ is a weakly prime ideal of S, the family of all $\overline{O}$, $-\overline{U}$ with $O \notin \mathcal{P}$ and $U \in \mathcal{P}$ has the finite intersection property. Indeed, if $O_1, \ldots, O_n \notin \mathcal{P}$ and $U_1, \ldots, U_m \in \mathcal{P}$, then, by Lemma 2.3,

$$\mathcal{P} \in r(O_1) \cap \ldots \cap r(O_n) \setminus r(U_1 \cup \ldots \cup U_m)$$

$$= \overline{r_1(O_1) \cap \ldots \cap r_1(O_n) \setminus r_1(U_1 \cup \ldots \cup U_m)}$$

and therefore $O_1 \cap \ldots \cap O_m \cap (-U_1) \cap \ldots \cap (-U_m) \neq \emptyset$. By the compactness of X, there exists $x \in X$ such that $x \in \bigcap \{\overline{O} \mid O \notin \mathcal{P}\} \cap \{-\overline{U} \mid U \in \mathcal{P}\}$, showing $\mathcal{P} = \xi(x)$.

c) One proves that ξ is an order-embedding in the same way that ξ is one-to-one and it remains to prove that ξ is continuous. A basic open set in $X(S)$ is of the form

$$\mathcal{O} = r(O_1) \cap \ldots \cap r(O_n) \setminus r(U)$$

whence $\xi^{-1}(\mathcal{O}) = \overline{O_1} \cap \ldots \cap \overline{O_n} \setminus \overline{U}$, which is open in X.

□

Theorem 2.5. *If (X_i, X_{1i}) is a Priestley structure for each $i \in I$, then $(\prod X_i, \prod X_{1i})$ is also a Priestley structure. Moreover a clopen decreasing set is of type 1 if and only if it can be written as a finite union of basic clopen decreasing sets of type 1, that is in the form*

$$\bigcap_{k=1}^{n} pr_{i_k}^{-1}(O_k)$$

where $O_k \in \mathcal{D}_1(X_{i_k})$ *for each* k.

Proof. The Priestley isomorphism ξ of Lemma 2.4 extends to a Priestley structure isomorphism by defining $X_1 = \xi^{-1}(X_1(S))$, and we shall prove that $X_1 = \prod X_{i1}$. Recall first that Stone duality gives an isomorphism $x \mapsto \varphi(x) = \{O \mid O \not\ni x\}$ from the space $Y = \prod X_{i1}$ onto the space of prime ideals of its dual semilattice S. This gives the following sequence of equivalences (using notations of the proof of Lemma 2.4) :

$$x \in \prod X_{i1},$$

$$\varphi(x) = \{O \in S \mid \overline{O} \ni x\} \text{ is prime in } S,$$

$$\xi(x) = \{O \in S \mid \overline{O} \ni x\} \ (= \varphi(x)) \in X_1(S),$$

$$x \in X_1.$$

To prove the second assertion, compose the isomorphisms $r : S \to \mathcal{D}_1 X(S)$ (given by Priestley duality) and $\xi^{-1} : \mathcal{D}_1 X(S) \to \mathcal{D}_1(X, X_1)$ (given by Lemma 2.4) to get an isomorphism $S \to \mathcal{D}_1(X, X_1)$ sending U on $\{x \mid \xi(x) \in r(U)\} = \{\xi \mid \xi(x) \not\ni U\} = \{x \mid x \in \overline{U}\} = \overline{U}$. Any $U \in S$ is a finite union of basic open sets in $\prod X_{i1}$, of the form

$$\bigcap_{k=1}^{n} pr_{i_k}^{-1}(U_k),$$

U_k compact open in (X_{1i}, τ_{1i}), and it remains to observe that this means $\overline{U_k}$ is of type 1 in $(X_i, \tau_i, \leq)$. $\square$

We end by a negative result. It is well-known that a bounded distributive lattice is boolean if and only if its prime spectrum is an antichain. This is no longer true if we consider bounded distributive semilattices, as shown in the following lines.

Proposition 2.6. *A bounded distributive semilattice is a boolean lattice if and only if its weak prime spectrum is an antichain. There exist bounded distributive semilattices whose prime spectrum is an antichain but whose weak prime spectrum is not an antichain.*

Proof. Let S be a bounded distributive semilattice whose weak prime spectrum $X(S)$ is an antichain. Since $X_1(S)$ is order-generating in $X(S)$ we have $X(S) = X_1(S)$, whence S is a lattice, and the result is true in this case.

To prove the second assertion, it suffices to consider any Priestley structure (P, P_1) as defined in Lemma 2.1 for which P_1 is an antichain and $\infty < x$ for finitely many $x \in P_1$. □

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Added in proof. While this paper was submitted, we have been aware that Guram Bezhanishvili and Ramon Jansana have found similar results that will appear in a paper untitled "Duality for distributive and implicative semi-lattices". Among other results they give a dualisation of usual homomorphisms between semi-lattices.

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