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Intersections of non quasi-analytic classes of ultradifferentiable functions

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Abstract

As in [8] and [9], we define the intersections $\mathcal{E}_{(\mathfrak{M})}(\Omega)$, $\mathcal{D}_{(\mathfrak{M})}(K)$ and $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ of non quasi-analytic classes by means of a matrix $\mathfrak{M}$. We prove that they differ from classical Beurling classes and that they coincide algebraically with the corresponding intersections of Roumieu classes. We next consider a few elementary properties and give a condition on $\mathfrak{M}$ under which these spaces are nuclear.

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1 Introduction

Intersections of non quasi-analytic classes have first been investigated by Chaumat and Chollet in [3] in the case $M_{j,p} = M_p^{a_j}$ where $(M_p)_{p\in\mathbb{N}_0}$ is a sequence with moderate growth and $(a_j)_{j\in\mathbb{N}}$ a sequence of positive numbers strictly decreasing to 0. They obtained a Whitney extension theorem, a Łojasiewicz theorem on regular situation, some theorem of division and preparation and a Whitney spectral theorem.

Later on Beaugendre studied extensively such intersections in [1] and [2] when the numbers $M_{j,p}$ are defined by means of a convex and increasing function Φ on $[0, +\infty[$ such that $\lim_{t\to\infty} \Phi(t)/t = \infty$. In particular he obtained extension results for Whitney jets and an explicit continuous linear extension map for Whitney jets.

We considered such intersections for general matrices $\mathfrak{M} = (M_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$ and obtained analytic and holomorphic extensions of Whitney jets in [8] and an explicit continuous linear extension map for Whitney jets in [9].

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In this paper we investigate general properties of the spaces $\mathcal{E}_{(\mathfrak{M})}(\Omega)$, $\mathcal{D}_{(\mathfrak{M})}(K)$ and $\mathcal{D}_{(\mathfrak{M})}(\Omega)$. We first prove that, under a mild condition for the matrix, they are never classical Beurling classes and that they coincide with the corresponding intersections of Roumieu classes. We next develop some elementary properties based on the following result: every bounded subset of $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ is a bounded subset of a space $\mathcal{E}^{(\boldsymbol{M}')}(\Omega)$ such that the canonical injection from $\mathcal{E}^{(\boldsymbol{M}')}(\Omega)$ into $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ is well defined, continuous and linear. We also prove that, if the compact subset K of $\mathbb{R}^n$ has the local displacement property, then $\mathcal{D}_{(\mathfrak{M})}(K)$ is a dense subspace of each one of the step spaces $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$. Finally we give a condition on $\mathfrak{M}$ under which these spaces are nuclear.

2 Notations

All functions we consider are complex valued and all vector spaces are $\mathbb{C}$-vector spaces. The euclidean norm of $x \in \mathbb{R}^n$ is denoted $|x|$. If f is a function defined on $A \subset \mathbb{R}^n$, then we set $\|f\|_A := \sup_{x\in A}|f(x)|$.

If E is a Hausdorff locally convex topological vector space (in short: a locally convex space), E' designates its topological dual endowed with the strong $\beta(E', E)$ topology. We refer to [4] and [7] for properties of locally convex spaces.

Whenever $\boldsymbol{m}$ is a sequence $(m_p)_{p\in\mathbb{N}_0}$ of real numbers, the notation $\boldsymbol{M}$ designates as usual the sequence $(M_p)_{p\in\mathbb{N}_0}$ where $M_p = m_0 \dots m_p$ for every $p \in \mathbb{N}_0$. Such a sequence is
(a) *normalized* if $m_0 = 1$ and $m_p \geq 1$ for every $p \in \mathbb{N}$;
(b) *non quasi-analytic* if $\sum_{p=0}^{\infty} 1/m_p < \infty$.

Let $\boldsymbol{m}$ be a normalized, increasing and non quasi-analytic sequence. Let moreover Ω be a non empty open subset of $\mathbb{R}^n$ and K be a compact subset of $\mathbb{R}^n$. Then as in [5], one can consider the following Beurling classes:
a) *the* (FS)-*space* $\mathcal{E}^{(\boldsymbol{M})}(\Omega)$: its elements are the $\mathcal{C}^\infty$-functions f on Ω such that

$$\|f\|_{H,h} := \sup_{\alpha\in\mathbb{N}_0^r} \frac{\|\mathrm{D}^\alpha f\|_H}{h^{|\alpha|}M_{|\alpha|}} < \infty$$

for every compact subset H of Ω and all constants $h > 0$, endowed with the fundamental system of semi-norms $\{\|\cdot\|_{H,h} : H \Subset \Omega, h > 0\}$;
b) *the* (FS)-*space* $\mathcal{D}^{(\boldsymbol{M})}(K)$: it is the topological subspace of $\mathcal{E}^{(\boldsymbol{M})}(\mathbb{R}^n)$, the elements of which have their support contained in K;
c) *the* (LFS)-*space* $\mathcal{D}^{(\boldsymbol{M})}(\Omega)$: it is the inductive limit of the spaces $\mathcal{D}^{(\boldsymbol{M})}(H)$ where H runs through the family of the compact subsets of Ω.

One also considers the following Roumieu classes which for the time being will not be endowed with locally convex topologies:
a) *the vector space* $\mathcal{E}^{\{\boldsymbol{M}\}}(\Omega)$: its elements are the $\mathcal{C}^\infty$-functions f on Ω such that, for every compact subset H of Ω, there are $A > 0$ and $h > 0$ such that $\|f\|_{H,h} < \infty$;
b) *the vector spaces* $\mathcal{D}^{\{\boldsymbol{M}\}}(K)$ *and* $\mathcal{D}^{\{\boldsymbol{M}\}}(\Omega)$ which have then a clear meaning.

As the sequence $\boldsymbol{m}$ is normalized and increasing, the sequence $\boldsymbol{M}$ is logarithmically convex; therefore it is well known (cf. [5]) that all these spaces are algebras.

From now on, throughout the paper $\mathfrak{m}$ designates a *semi-regular* matrix

$$\mathfrak{m} = (m_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0},$$

i.e. a matrix such that, for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j = (m_{j,p})_{p\in\mathbb{N}_0}$ is normalized, increasing, non quasi-analytic and such that
(a) $m_{j,p} \geq m_{j+1,p}$ for every $p \in \mathbb{N}_0$,
(b) $\lim_{p\to\infty} m_{j+1,p}/m_{j,p} = 0$.

Then, of course, $\boldsymbol{M}_j$ designates the sequence $(M_{j,p})_{p\in\mathbb{N}_0}$ for every $j \in \mathbb{N}_0$ and $\mathfrak{M}$ the matrix $(M_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$.

This allows to introduce the following vector spaces

$$\mathcal{E}_{(\mathfrak{M})}(\Omega) = \bigcap_{j\in\mathbb{N}} \mathcal{E}^{(\boldsymbol{M}_j)}(\Omega); \quad \mathcal{E}_{\{\mathfrak{M}\}}(\Omega) = \bigcap_{j\in\mathbb{N}} \mathcal{E}^{\{\boldsymbol{M}_j\}}(\Omega);$$

$$\mathcal{D}_{(\mathfrak{M})}(\Omega) = \bigcap_{j\in\mathbb{N}} \mathcal{D}^{(\boldsymbol{M}_j)}(\Omega); \quad \mathcal{D}_{\{\mathfrak{M}\}}(\Omega) = \bigcap_{j\in\mathbb{N}} \mathcal{D}^{\{\boldsymbol{M}_j\}}(\Omega);$$

$$\mathcal{D}_{(\mathfrak{M})}(K) = \bigcap_{j\in\mathbb{N}} \mathcal{D}^{(\boldsymbol{M}_j)}(K); \quad \mathcal{D}_{\{\mathfrak{M}\}}(K) = \bigcap_{j\in\mathbb{N}} \mathcal{D}^{\{\boldsymbol{M}_j\}}(K).$$

Of course, we then define
a) *the* (FS)*-space* $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ as the projective limit of the spaces $\mathcal{E}^{(\boldsymbol{M}_j)}(\Omega)$, i.e. we endow it with the fundamental system of the semi-norms $\|\cdot\|_{H,h,j}$ defined by

$$\|f\|_{H,h,j} := \sup_{\alpha\in\mathbb{N}_0^n} \frac{\|\mathrm{D}^\alpha f\|_H}{h^{|\alpha|} M_{j,|\alpha|}}$$

where H is a compact subset of Ω, $h > 0$ and $j \in \mathbb{N}$;
b) *the* (FS)*-space* $\mathcal{D}_{(\mathfrak{M})}(K)$ as a topological subspace of $\mathcal{E}_{(\mathfrak{M})}(\mathbb{R}^n)$;
c) *the* (LFS)*-space* $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ as the inductive limit of the spaces $\mathcal{D}_{(\mathfrak{M})}(H)$ where H runs through the family of the compact subsets of Ω.

If g belongs to $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ (resp. $\mathcal{D}_{(\mathfrak{M})}(\Omega)$), then the multiplication map $M_g\colon f \mapsto gf$ is a continuous linear map from $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ into itself as well as from $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ into itself (resp. from $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ into $\mathcal{D}_{(\mathfrak{M})}(\Omega)$) (cf. [5]).

3 Equality of vector spaces

Proposition 3.1 *For every semi-regular matrix $\mathfrak{M}$, the following equalities of vector spaces hold*

$$\mathcal{E}_{(\mathfrak{M})}(\Omega) = \mathcal{E}_{\{\mathfrak{M}\}}(\Omega); \quad \mathcal{D}_{(\mathfrak{M})}(\Omega) = \mathcal{D}_{\{\mathfrak{M}\}}(\Omega); \quad \mathcal{D}_{(\mathfrak{M})}(K) = \mathcal{D}_{\{\mathfrak{M}\}}(K).$$

Proof. It clearly suffices to establish the first one of these equalities.

On one hand, the inclusion $\mathcal{E}_{(\mathfrak{M})}(\Omega) \subset \mathcal{E}_{\{\mathfrak{M}\}}(\Omega)$ is a direct consequence of the fact that, for every $j \in \mathbb{N}$, we certainly have $\mathcal{E}^{(\boldsymbol{M}_j)}(\Omega) \subset \mathcal{E}^{\{\boldsymbol{M}_j\}}(\Omega)$.

To establish the other inclusion, it suffices to prove that, for every $j \in \mathbb{N}$, every element f of $\mathcal{E}^{\{\boldsymbol{M}_{j+1}\}}(\Omega)$ belongs to $\mathcal{E}^{(\boldsymbol{M}_j)}(\Omega)$.

Let H be a compact subset of Ω and $h > 0$ be fixed. As f belongs to $\mathcal{E}^{\{\boldsymbol{M}_{j+1}\}}(\Omega)$, there are $A_{j+1} > 0$ and $h_{j+1} > 0$ such that

$$\|\mathrm{D}^\alpha f\|_H \leq A_{j+1} h_{j+1}^{|\alpha|} M_{j+1,|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n.$$

As $\mathfrak{m}$ is semi-regular, there is an integer $p_0 \in \mathbb{N}$ such that

$$\frac{m_{j+1,p}}{m_{j,p}} \leq \frac{h}{h_{j+1}}, \quad \forall p \geq p_0;$$

this leads to a constant $B_j > 0$ such that, for every $p \geq p_0$,

$$\begin{aligned} \frac{M_{j+1,p}}{M_{j,p}} &= \frac{M_{j+1,p_0}}{M_{j,p_0}} \frac{m_{j+1,p_0+1}}{m_{j,p_0+1}} \cdots \frac{m_{j+1,p}}{m_{j,p}} \\ &\leq \frac{M_{j+1,p_0}}{M_{j,p_0}} \left(\frac{h}{h_{j+1}}\right)^{p-p_0} = B_j \left(\frac{h}{h_{j+1}}\right)^p. \end{aligned}$$

Therefore, for every $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \geq p_0$, we get

$$\|\mathrm{D}^\alpha f\|_H \leq A_{j+1} B_j h^{|\alpha|} M_{j,|\alpha|}.$$

Moreover, if we set

$$C_j := \sup \left\{ \left(\frac{h_{j+1}}{h}\right)^p \frac{M_{j+1,p}}{M_{j,p}} : p < p_0 \right\},$$

we obtain

$$\|\mathrm{D}^\alpha f\|_H \leq A_{j+1} C_j h^{|\alpha|} M_{j,|\alpha|}$$

for every $\alpha \in \mathbb{N}_0^r$ such that $|\alpha| < p_0$ and we conclude at once.∎

4 Autonomy of the spaces $\mathcal{E}_{(\mathfrak{M})}(\Omega)$

The following result establishes the autonomy of the spaces $\mathcal{E}_{(\mathfrak{M})}(\Omega)$. The idea of its proof goes back to the one of Proposition 24 of [8]; for the sake of clarity, we give a full proof.

Proposition 4.1 *Let the matrix* $\mathfrak{m}$ *verify*

$$\frac{m_{j,p}}{p} \leq \frac{m_{j,p+1}}{p+1}, \quad \forall j, p \in \mathbb{N},$$

and suppose that, for every $j \in \mathbb{N}$*, there is* $A(j) > 0$ *such that*

$$M_{j+1,p} M_{j+1,p+1} \leq A(j)^{p+1}(p+1)! M_{j,p}, \quad \forall p \in \mathbb{N}_0.$$

Then, for any normalized, increasing and non quasi-analytic sequence $\boldsymbol{r}$*, the vector spaces* $\mathcal{E}_{(\mathfrak{M})}(]-1,1[)$ *and* $\mathcal{E}^{(\boldsymbol{R})}(]-1,1[)$ *differ.*

Proof. Let us suppose that there are such a matrix $\mathfrak{m}$ and a sequence $\boldsymbol{r}$ for which the equality of these two spaces holds.

By Theorem 13 of [8], for every sequence $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}_0}$ of complex numbers such that

$$\sup_{p\in\mathbb{N}_0} \frac{|a_p|}{h^p M_{j,p}} < \infty, \quad \forall h > 0, j \in \mathbb{N},$$

there is a $\mathcal{C}^\infty$-function f on $\mathbb{R}$, with compact support and such that

$$\sup_{p\in\mathbb{N}_0} \frac{\|\mathrm{D}^p f\|_{\mathbb{R}}}{h^p M_{j,p}} < \infty, \quad \forall h > 0, j \in \mathbb{N}.$$

As we may suppose $\mathrm{supp}(f) \subset]-1,1[$, we get

$$f \in \mathcal{E}_{(\mathfrak{M})}(]-1,1[) = \mathcal{E}^{(\boldsymbol{R})}(]-1,1[)$$

which leads to the fact that, for such sequences $\boldsymbol{a}$, we also have

$$\sup_{p\in\mathbb{N}_0} \frac{|a_p|}{h^p R_p} < \infty, \quad \forall h > 0.$$

With the notations of [8], this means that we have $\widehat{\Lambda}_{(\mathfrak{M})} \subset \widehat{\Lambda}_{(\mathfrak{R})}$ where the matrix $\mathfrak{r} = (r_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$ is defined by $r_{j,p} = r_p$ for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$.

By hypothesis, we of course have

$$\frac{M_{j,p}}{2^p M_{j+1,p}} \geq \frac{M_{j+1,p+1}}{2^{p+1} A(j)^{p+1}(p+1)!}, \quad \forall j \in \mathbb{N}, p \in \mathbb{N}_0.$$

Moreover, for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j$ is normalized, increasing and non quasi-analytic; this implies $p/m_{j,p} \to 0$ hence

$$\lim_p \left(\frac{(p+1)!}{M_{j+1,p+1}} \right)^{1/(p+1)} = 0, \quad \forall j \in \mathbb{N}.$$

Putting these two informations together leads to

$$\liminf_p \frac{M_{j,p}}{2^p M_{j+1,p}} > 1, \forall j \in \mathbb{N}.$$

Now we apply Proposition 22 of [8] and get the existence of an integer $c > 1$ such that $M_{c,p} \leq R_{1,p} = R_p$ for every integer $p \geq c$. This implies

$$\mathcal{E}^{(\boldsymbol{M}_c)}(]-1,1[) \subset \mathcal{E}^{(\boldsymbol{R})}(]-1,1[) = \mathcal{E}_{(\mathfrak{M})}(]-1,1[)$$

hence $\mathcal{E}^{(\boldsymbol{M}_j)}(]-1,1[) = \mathcal{E}^{(\boldsymbol{R})}(]-1,1[)$ for every $j \geq c$.

However, the hypothesis leads directly to

$$\lim_p \left(\frac{M_{c+1,p}}{M_{c+2,p}} \right)^{1/p} \geq \lim_p A(c+1)^{-(p+1)/p} \left(\frac{M_{c+2,p+1}}{(p+1)!} \right)^{1/p} = \infty.$$

So the result ([6], 6.7.I) provides the existence of a function

$$f \in \mathcal{E}^{\{\boldsymbol{M}_{c+1}\}}(]-1,1[) \setminus \mathcal{E}^{\{\boldsymbol{M}_{c+2}\}}(]-1,1[)$$

hence a contradiction.∎

5 Elementary properties

We start with the following lemma, the proof of which goes back to ideas of Chaumat and Chollet [3].

Lemma 5.1 *For every bounded subset B of $\mathcal{E}_{(\mathfrak{M})}(\Omega)$, there is a normalized, increasing and non quasi-analytic sequence $\boldsymbol{m}' = (m'_p)_{p \in \mathbb{N}_0}$ such that, for every $j \in \mathbb{N}$, there is a constant $C(j) > 0$ such that $m'_p \leq C(j) m_{j,p}$ for every $p \in \mathbb{N}_0$ and such that B is a bounded subset of $\mathcal{E}^{(\boldsymbol{M}')}(\Omega)$.*

Proof. Let $(K_n)_{n \in \mathbb{N}}$ be a fundamental sequence of compact subsets of Ω and set for every $j \in \mathbb{N}$

$$k_j := \sup_{f \in B} \sup_{\alpha \in \mathbb{N}_0^n} \frac{2^{2j|\alpha|} \left\| \mathrm{D}^\alpha f \right\|_{K_j}}{M_{j,|\alpha|}}.$$

Now we define the sequences $(n_q)_{q\in\mathbb{N}_0}$ and $(h_q)_{q\in\mathbb{N}_0}$ by means of the following recursion:
(a) we set $n_0 = h_0 = 0$;
(b) once n_{q-1} and h_{q-1} are determined, n_q is the first integer for which

$$n_q > h_{q-1}, \quad \sum_{p_q}^{\infty} \frac{1}{m_{q+1,p}} \leq 2^{-q} \quad \text{and} \quad k_{q+1}2^{-n_q} \leq 1$$

and then h_q is the first integer such that $m_{q,n_q} < m_{q+1,h_q}$. In particular, as the sequence $(m_{q+1,p})_{p\in\mathbb{N}_0}$ is increasing and as $m_{q,n_q} \geq m_{q+1,n_q}$, we certainly have $n_q < h_q$.

This allows us to introduce the sequence $\boldsymbol{m'} = (m'_p)_{p\in\mathbb{N}_0}$ as follows: we set $m'_0 = 1$ and for every $p \in \mathbb{N}$

$$m'_p = \begin{cases} m_{q,p} & \text{if} \quad p \in \{h_{q-1}, \ldots, n_q\}, \\ m_{q,n_q} & \text{if} \quad p \in \{n_q + 1, \ldots, h_q - 1\}. \end{cases}$$

It is clear that this sequence $\boldsymbol{M'}$ is normalized and increasing. It also is non quasi-analytic since we successively have

$$\sum_{p=n_1+1}^{\infty} \frac{1}{m'_p} = \sum_{q=1}^{\infty} \left(\sum_{p=n_q+1}^{h_q-1} \frac{1}{m_{q,n_q}} + \sum_{p=h_q}^{n_{q+1}} \frac{1}{m_{q+1,p}} \right)$$
$$\leq \sum_{q=1}^{\infty} \sum_{p=n_q+1}^{\infty} \frac{1}{m_{q+1,p}} \leq 1.$$

Moreover for every $j \in \mathbb{N}$, the existence of the constant $C(j)$ is a direct consequence of the fact that, for every $p \geq n_j$, we have $m'_p \leq m_{j,p}$.

To conclude, we now prove that B is a bounded subset of $\mathcal{E}^{(\boldsymbol{M'})}(\Omega)$. Let $\|\cdot\|_{K,h}$ be any fundamental continuous semi-norm of $\mathcal{E}^{(\boldsymbol{M'})}(\Omega)$. There is then $k \in \mathbb{N}$ such that $2^k > h^{-1}$ and $K \subset K_k$. So, for every $\alpha \in \mathbb{N}_0^r$ such that $|\alpha| > n_k$, there is a unique integer $j > k$ such that $n_{j-1} < |\alpha| \leq n_j$; as this implies $m_{j,p} \leq m'_p$ for every $p \in \mathbb{N}_0$ verifying $p \leq |\alpha|$, hence $M_{j,|\alpha|} \leq M_{|\alpha|}$, we get

$$\frac{\|\mathrm{D}^\alpha f\|_K}{h^{|\alpha|} M'_{|\alpha|}} \leq \frac{2^{k|\alpha|} \|\mathrm{D}^\alpha f\|_{K_j}}{M_{j,|\alpha|}} \leq 2^{-j|\alpha|} k_j \leq 2^{-n_{j-1}} k_j \leq 1,$$

for every $f \in B$. The conclusion is then immediate.∎

Proposition 5.2 a) *For every compact subsets H and K of Ω such that $\emptyset \neq K \subset H^\circ$, there is $f \in \mathcal{D}_{(\mathfrak{M})}(H)$, $f \geq 0$, identically 1 on a neighbourhood of K.*

b) *If $\{\Omega_j : j \in J\}$ is a finite open cover of the compact subset K of Ω, then there are functions $\varphi_j \in \mathcal{D}_{(\mathfrak{M})}(\Omega)$ such that $\varphi_j \geq 0$ and $\sum_{j\in J} \varphi_j \equiv 1$ on a neighbourhood of K.*

c) *For every open cover $\mathcal{O}$ of Ω, there is a $\mathcal{D}_{(\mathfrak{M})}(\Omega)$-partition of unity on Ω, subordinate to $\mathcal{O}$.*

Proof. As the function χ_Ω belongs to $\mathcal{E}_{(\mathfrak{M})}(\Omega)$, we may consider the sequence $\boldsymbol{m}'$ that Lemma 5.1 associates to the bounded subset $\{\chi_\Omega\}$.

As the existence of the constants $C(j)$ implies that $\mathcal{E}^{(\boldsymbol{M}')}(\Omega)$ is contained in $\mathcal{E}_{(\mathfrak{M})}(\Omega)$, the Denjoy-Carleman-Mandelbrojt theorem leads directly to the conclusion: cf. ([5], Theorem 4.2) in the case a), ([5], Lemma 5.1) in the case b) and ([5], Proposition 5.2) in the case c).∎

Proposition 5.3 *The vector space $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ is dense in $\mathcal{E}_{(\mathfrak{M})}(\Omega)$.*

Proof. This is standard.

Let $(K_n)_{n\in\mathbb{N}}$ be a fundamental sequence of compact subsets of Ω such that $K_n \subset K_{n+1}^\circ$ for every $n \in \mathbb{N}$. By Proposition 5.2 (a), there is then a sequence $(\varphi_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, φ_n belongs to $\mathcal{D}_{(\mathfrak{M})}(K_{n+1})$ and is identically 1 on a neighbourhood of K_n. To conclude, it suffices to check that, for every $f \in \mathcal{E}_{(\mathfrak{M})}(\Omega)$, the sequence $(f\varphi_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{E}_{(\mathfrak{M})}(\Omega)$.∎

These properties imply consequences about the $\mathfrak{M}$-*distributions on* Ω, i.e. the elements of the topological dual of $\mathcal{D}_{(\mathfrak{M})}(\Omega)$.

Proposition 5.4 a) *Every $\mathfrak{M}$-distribution on Ω has a support.*

b) *Every distribution on Ω is a $\mathfrak{M}$-distribution.*

c) *A $\mathfrak{M}$-distribution has a compact support if and only if it has a continuous linear extension on $\mathcal{E}_{(\mathfrak{M})}(\Omega)$.*

Proof. a) is a direct consequence of the part (b) of the Proposition 5.2.

b) is trivial since the canonical injection from $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ into $\mathcal{D}(\Omega)$ is a well defined continuous linear map.

c) is standard. Let $u \in \mathcal{D}_{(\mathfrak{M})}(\Omega)'$ have a compact support K. We then choose a compact subset H of Ω such that $K \subset H^\circ$ and next an element ψ of $\mathcal{D}_{(\mathfrak{M})}(H^\circ)$ identically 1 on a neighbourhood of K. We then have $u(\varphi) = u(\varphi\psi)$ for every $\varphi \in \mathcal{D}_{(\mathfrak{M})}(\Omega)$. Now we define the linear functional v on $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ by $v(f) = u(f\psi)$. As we have $v(\varphi) = u(\varphi\psi) = u(\varphi)$ for every $\varphi \in \mathcal{D}_{(\mathfrak{M})}(\Omega)$, v is a linear extension of u to the Fréchet space $\mathcal{E}_{(\mathfrak{M})}(\Omega)$. Now if $(f_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ converging to 0, $(f_n\psi)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{D}_{(\mathfrak{M})}(H)$ converging to 0 hence such that $u(f_n\psi) \to 0$. Hence the conclusion.∎

6 Approximation

Proposition 6.1 *The set of the restrictions of the polynomials to Ω is a dense vector subspace of $\mathcal{E}_{(\mathfrak{M})}(\Omega)$.*

Proof. This is clear since we know by [5] that the set of the restrictions of the polynomials to Ω is a dense vector subspace of each one of the spaces $\mathcal{E}^{(\boldsymbol{M}_j)}(\Omega)$.∎

Definition. By $\mathcal{E}^{(p!)}(\mathbb{R}^n)$, we designate the space of the $\mathcal{C}^\infty$-functions f on $\mathbb{R}^n$ such that, for every $h > 0$ and non void compact subset K of $\mathbb{R}^n$,

$$|f|_{K,h} := \sup_{\alpha \in \mathbb{N}_0^n} \frac{\|\mathrm{D}^\alpha f\|_K}{h^{|\alpha|}\alpha!} < \infty$$

endowed with the system $\{|\cdot|_{K,h} : K \Subset \mathbb{R}^n, h > 0\}$ of semi-norms; it clearly is a Fréchet space.

We denote by $\mathcal{H}(\mathbb{C}^n)$ the Fréchet space of the entire functions on $\mathbb{C}^n$ endowed with the topology of uniform convergence on the compact subsets.

The following result is easily obtained by classical holomorphy arguments.

Proposition 6.2 *The restriction map*

$$\Gamma\colon \mathcal{H}(\mathbb{C}^n) \to \mathcal{E}^{(p!)}(\mathbb{R}^n); \quad f \mapsto f|_{\mathbb{R}^n}$$

is a well defined isomorphism.∎

Proposition 6.3 *The restriction map*

$$R_\Omega\colon \mathcal{E}^{(p!)}(\mathbb{R}^n) \to \mathcal{E}_{(\mathfrak{M})}(\Omega); \quad f \mapsto f|_\Omega$$

is well defined, continuous, linear and injective.

Proof. For every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j$ is increasing, normalized and non quasi-analytic; this implies $\lim_p p/m_{j,p} = 0$ hence there is $B_j > 0$ such that $p! \leq B_j M_{j,p}$ for every $p \in \mathbb{N}_0$. Therefore for every $f \in \mathcal{E}^{(p!)}(\mathbb{R}^n)$, compact subset K of $\mathbb{R}^n$ and $h > 0$, we get

$$\|\mathrm{D}^\alpha f\|_K \leq |f|_{K,h}\, h^{|\alpha|}\alpha! \leq B_j\, |f|_{K,h}\, h^{|\alpha|} M_{j,|\alpha|}$$

for every $\alpha \in \mathbb{N}_0^n$.∎

Proposition 6.4 *If the compact subsets K and H of $\mathbb{R}^n$ are such that $H \subset K^\circ$, then, for every $j \in \mathbb{N}$, the closure of $\mathcal{D}_{(\mathfrak{M})}(K)$ in $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$ contains $\mathcal{D}^{(\boldsymbol{M}_j)}(H)$.*

Proof. Let f belong to $\mathcal{D}^{(\boldsymbol{M}_j)}(H)$ and consider any continuous semi-norm $\|.\|_{K,j,h}$ on $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$ and $\varepsilon > 0$.

We first choose $\varphi \in \mathcal{D}_{(\mathfrak{M})}(K)$ identically 1 on a neighbourhood of H. Setting

$$\psi_m(x) = \pi^{-n/2} m^n e^{-m^2|x|^2}, \quad \forall x \in \mathbb{R}^n, m \in \mathbb{N},$$

we know that the sequence $(f * \psi_m)_{m\in\mathbb{N}}$ converges to f in $\mathcal{E}^{(\boldsymbol{M}_j)}(\mathbb{R}^n)$ hence that the sequence $((f * \psi_m).\varphi)_{m\in\mathbb{N}}$ converges to $f\varphi = f$ in $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$. In particular, there is $m_0 \in \mathbb{N}$ such that $\|f - (f * \psi_m).\varphi\|_{K,j,h} \leq \varepsilon/2$.

As $f * \psi_{m_0}$ has a holomorphic extension on $\mathbb{C}^n$, it belongs to $\mathcal{E}^{(p!)}(\mathbb{R}^n)$ and there is a sequence $(P_l)_{l\in\mathbb{N}}$ of polynomials converging to $f * \psi_{m_0}$ in $\mathcal{E}^{(p!)}(\mathbb{R}^n)$. Therefore the sequence $(P_l\varphi)_{l\in\mathbb{N}}$ converges to $(f * \psi_{m_0}).\varphi$ in $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$ and we can conclude.∎

To obtain cases when $\mathcal{D}_{(\mathfrak{M})}(K)$ is dense in $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$ we need some information and a definition.

Notation. Given $b \in \mathbb{R}^n$ and a function f on $\mathbb{R}^n$, $\tau_b f$ designates the function defined on $\mathbb{R}^n$ by $\tau_b f(.) = f(. - b)$.

Proposition 6.5 *For every $b \in \mathbb{R}^n$, τ_b is a well defined continuous linear map from $\mathcal{E}_{(\mathfrak{M})}(\mathbb{R}^n)$ into itself.*

Moreover we have $\lim_{b\to 0} \tau_b f = f$ for every $f \in \mathcal{E}_{(\mathfrak{M})}(\mathbb{R}^n)$.

Proof. The first part is trivial.

Let $\|.\|_{K,h,j}$ be any continuous semi-norm on $\mathcal{E}_{(\mathfrak{M})}(\mathbb{R}^n)$ and $\varepsilon > 0$ be given. We first choose a compact subset H of $\mathbb{R}^n$ such that $K \subset H^\circ$ and set $\delta = d(K, \mathbb{R}^n \setminus H^\circ)$. We then choose $m \in \mathbb{N}$ such that $2^{-m} \|f\|_{H,j,h} \leq \varepsilon/2$.

On the one hand, for every $x \in K$, $b \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ such that $|b| \leq \delta$ and $|\alpha| \geq m$, we immediately get

$$\frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(x-b)|}{(2h)^{|\alpha|} M_{|\alpha|}} \leq \frac{2}{2^m} \|f\|_{H,j,h} \leq \varepsilon.$$

On the other hand $\{\mathrm{D}^\alpha f\colon\ |\alpha| < m\}$ is a finite set of continuous functions on the compact set H hence is uniformly equicontinuous on H.

These two informations put together lead directly to the conclusion.∎

Definition. A subset B of $\mathbb{R}^n$ *has the local displacement property* if every $x \in B$ has a neighbourhood W such that, for every $\varepsilon > 0$, there is $a \in \mathbb{R}^n$ such that $|a| \leq \varepsilon$ and $a + B \cap W \subset B^\circ$.

If $B_1, \ldots, B_q$ are finitely many closed balls in $\mathbb{R}^n$ such that $B_j \cap B_k \neq \emptyset$ implies $B_j^\circ \cap B_k^\circ \neq \emptyset$, one can check that their union has the local displacement property. Therefore every open subset of $\mathbb{R}^n$ has an exhaustion $(K_l)_{l\in\mathbb{N}}$ made of compact sets having the local displacement property and such that $K_l \subset K_{l+1}^\circ$ for every $l \in \mathbb{N}$.

Proposition 6.6 *If the compact subset K of $\mathbb{R}^n$ has the local displacement property, then, for every $j \in \mathbb{N}$, $\mathcal{D}_{(\mathfrak{M})}(K)$ is a dense vector subspace of $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$.*

Proof. Let f be an element of $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$ and let us consider any continuous semi-norm $\|.\|_{K,j,h}$ on $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$ and $\varepsilon > 0$.

As K has the local displacement property, it has a finite open cover $\{W_1, \ldots, W_q\}$ such that, for every $\delta > 0$ and $l \in \{1, \ldots, q\}$, there is $u_l \in \mathbb{R}^n$ such that $|u_l| \leq \delta$ and $u_l + K \cap W_l \subset K^\circ$. We then choose $\varphi_1 \in \mathcal{D}_{(\mathfrak{M})}(W_1)$, $\ldots$, $\varphi_q \in \mathcal{D}_{(\mathfrak{M})}(W_q)$ such that $\varphi_1, \ldots, \varphi_q \geq 0$ and $\sum_{l=1}^q \varphi_l \equiv 1$ on a neighbourhood of K.

We next set $f_l = f\varphi_l$ for $l = 1, \ldots, q$. From $f \in \mathcal{D}^{(\boldsymbol{M}_j)}(K)$ and $\varphi_l \in \mathcal{D}_{(\mathfrak{M})}(W_l)$, we obtain $f_l \in \mathcal{D}^{(\boldsymbol{M}_j)}(K)$. Therefore there is $\eta > 0$ such that $\|\tau_b f_l - f_l\|_{\mathbb{R}^n,j,h} \leq \varepsilon/(2q)$ for every $l = 1, \ldots, q$ whenever $|b| \leq \eta$.

Finally for every $l \in \{1, \ldots, q\}$, we choose $z_l \in \mathbb{R}^n$ such that $|z_l| \leq \eta$ and $z_l + (K \cap W_l) \subset K^\circ$. Therefore $\tau_{z_l} f_l$ belongs to $\mathcal{D}^{(\boldsymbol{M}_j)}(H_l)$ for some compact subset H_l such that $H_l \subset K^\circ$. So, by Proposition 6.4, there is $g_l \in \mathcal{D}_{(\mathfrak{M})}(K)$ such that $\|\tau_{z_l} f_l - g_l\|_{\mathbb{R}^n,j,h} \leq \varepsilon/(2q)$.

Hence the conclusion since this leads to $\|f - \sum_{l=1}^q g_l\|_{K,j,h} \leq \varepsilon$ with $\sum_{l=1}^q g_l \in \mathcal{D}_{(\mathfrak{M})}(K)$.∎

7 Regularity of $\mathfrak{M}$ and the maps D^β

Definition. Let us say that $\mathfrak{m}$ (or equivalently $\mathfrak{M}$) is *regular* if, for every $j \in \mathbb{N}$, there are constants $A(j) > 1$ and $H(j) > 1$ such that

$$M_{j+1,p+1} \leq A(j)H(j)^p M_{j,p}, \quad \forall p \in \mathbb{N}.$$

Proposition 7.1 *If $\mathfrak{M}$ is regular, then, for every $\beta \in \mathbb{N}_0^r$, D^β is a continuous linear map from $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ into itself as well as from $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ into itself.*

Proof. It clearly suffices to establish the statement in the case $|\beta| = 1$.

Let $\|\cdot\|_{K,h,j}$ be any fundamental continuous semi-norm on $\mathcal{E}_{(\mathfrak{M})}(\Omega)$. For every $\alpha \in \mathbb{N}_0^n$, a direct calculation shows

$$\left\|\mathrm{D}^{\alpha+\beta} f\right\|_K \leq \|f\|_{K,h/H(j),j+1} \frac{A(j)}{H(j)} h^{|\alpha|+1} M_{j,|\alpha|}$$

hence

$$\left\|\mathrm{D}^{\beta} f\right\|_{K,h,j} = \sup_{\alpha \in \mathbb{N}_0^r} \frac{\left\|\mathrm{D}^{\beta+\alpha} f\right\|_K}{h^{|\alpha|} M_{j,|\alpha|}} \leq \frac{A(j)}{H(j)} h \|f\|_{K,h/H(j),j+1}$$

and we conclude at once.∎

8 Regularity of $\mathfrak{M}$ and nuclearity

Notation. a) For every $\beta \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we set $\langle \beta, x\rangle := \beta_1 x_1 + \cdots + \beta_n x_n$.

b) For every β, $\gamma \in \mathbb{N}_0^n$ with $\beta \neq 0$, we set

$$\beta^2 := \beta_{i_1}^2 \ldots \beta_{i_s}^2$$
$$\beta^\gamma := \beta_{i_1}^{\gamma_{i_1}} \ldots \beta_{i_s}^{\gamma_{i_s}}$$

where $\beta_{i_1}, \ldots, \beta_{i_s}$ are the non zero elements of the finite sequence β.

c) We denote by $\mathcal{P}_{(\mathfrak{M})}(\mathbb{R}^n)$ the subspace of $\mathcal{E}_{(\mathfrak{M})}(\mathbb{R}^n)$, the elements of which are 2π-periodic in each component. Of course, its topology is also given by the system of norms $\{|||\cdot|||_{h,j} : h > 0, j \in \mathbb{N}\}$ with

$$|||f|||_{h,j} := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in [-\pi,\pi]^n} \frac{|\mathrm{D}^\alpha f(x)|}{h^{|\alpha|} M_{j,|\alpha|}}.$$

Proposition 8.1 *If $\mathfrak{M}$ is regular, then $\mathcal{P}_{(\mathfrak{M})}(\mathbb{R}^n)$ is a Fréchet nuclear space.*

Proof. Given the fundamental continuous semi-norm $|||\cdot|||_{h,j}$ on the space $\mathcal{P}_{(\mathfrak{M})}(\mathbb{R}^n)$, let Y (resp. X) be the Banach completion of the normed space $\mathcal{P}_{(\mathfrak{M})}(\mathbb{R}^n)$ endowed with $|||\cdot|||_{h,j}$ (resp. $|||\cdot|||_{h/H(j)^{2n},j+2n}$). Of course the elements of X and Y are $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$ which are 2π-periodic in each component. To conclude we just need to prove that the canonical injection from X into Y is nuclear.

For every $\beta \in \mathbb{N}_0^n$, we define the functional u_β on $\mathcal{P}_{(\mathfrak{M})}(\mathbb{R}^n)$ by

$$\langle f, u_\beta\rangle := \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} f(t) e^{-i\langle\beta,t\rangle}\, dt, \quad \forall f \in \mathcal{P}_{(\mathfrak{M})}(\mathbb{R}^n).$$

It is clear that u_β belongs to X'. Moreover it is well known that, for any $f \in \mathcal{P}_{(\mathfrak{M})}(\mathbb{R}^n)$, the series $\sum_{\beta\in\mathbb{N}_0^n} \langle f, u_\beta\rangle\, e^{i\langle\beta,x\rangle}$ converges in $\mathcal{C}^\infty(\mathbb{R}^n)$ to f. In fact, we have a lot more.

For this purpose, we first remark that, by repeated use of the regularity of $\mathfrak{M}$ where we may suppose the sequence $(H(j))_{j\in\mathbb{N}}$ increasing, we obtain for any $s \in \{1,\ldots,n\}$

$$\begin{aligned} M_{j+2n,p+2s} &\le A(j+2n-1)\ldots A(j+2n-2s)\cdot \\ &\quad \cdot H(j+2n-1)^{p+2s-1}\ldots H(j+2n-2s)^p M_{j+2n-2s,p} \\ &\le \left(\prod_{k=0}^{2n-1} A(j+k)\right).H(j+2n)^{2np+n(2n-1)} M_{j,p}; \end{aligned}$$

i.e. for every $j\in\mathbb{N}$, there are constants $A'(j)>1$ and $H'(j)>1$ such that

$$M_{j+2n,p+2s} \le A'(j)H'(j)^{2np} M_{j,p}, \quad \forall p\in\mathbb{N}, s\in\{1,\ldots,n\}.$$

For any $\beta\in\mathbb{N}_0^n$ different from 0, there is $\gamma\in\mathbb{N}_0^n$ such that

$$\left|\left|\left|e^{i\langle\beta,\cdot\rangle}\right|\right|\right|_{h,j} \le 2 \sup_{x\in[-\pi,\pi]^n} \frac{\left|\mathrm{D}^\gamma e^{i\langle\beta,x\rangle}\right|}{h^{|\gamma|}M_{j,|\gamma|}} = 2\frac{\beta^\gamma}{h^{|\gamma|}M_{j,|\gamma|}}$$

and we may impose $\gamma_p = 0$ whenever $\beta_p = 0$. Now we define $\gamma'\in\mathbb{N}_0^n$ as follows: $\gamma'_p = 0$ if $\gamma_p = 0$ and $\gamma'_p = \gamma_p + 2$ otherwise. Then integrating by parts leads to

$$\left|\int_{[-\pi,\pi]^n} f(t)e^{-i\langle\beta,t\rangle}\,dt\right| = \frac{1}{\beta^{\gamma'}}\left|\int_{[-\pi,\pi]^n} \mathrm{D}^{\gamma'} f(t).e^{-i\langle\beta,t\rangle}\,dt\right|.$$

If s is the number of the non zero components of γ, then, for every $t\in[-\pi,\pi]^n$, we have

$$\left|\mathrm{D}^{\gamma'} f(t)\right| \le |||f|||_{h/H'(j)^{2n},j+2n}\left(\frac{h}{H'(j)^{2n}}\right)^{|\gamma|+2s} M_{j+2n,|\gamma|+2s}.$$

A direct use of these informations leads to

$$\left|\left|\left|\langle f,u_\beta\rangle\, e^{i\langle\beta,.\rangle}\right|\right|\right|_{h,j} \le 2A'(j)\frac{h^{2s}}{H'(j)^{4ns}}\,|||f|||_{h/H'(j)^{2n},j+2n}\,\frac{1}{\beta^2}.$$

Hence the conclusion since the series $\sum_{\beta\in\mathbb{N}_0^n} 1/\beta^2$ converges.∎

Theorem 8.2 *If $\mathfrak{M}$ is regular,*
(a) *the Fréchet space $\mathcal{D}_{(\mathfrak{M})}(K)$ is nuclear;*
(b) *the* (LF)*-space $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ is nuclear;*
(c) *the Fréchet space $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ is nuclear.*

Proof. (a) There is $a > 0$ such that $H := aK$ is contained in $[-\pi, \pi]^n$. Of course, the map

$$T_a : \mathcal{D}_{(\mathfrak{M})}(K) \to \mathcal{D}_{(\mathfrak{M})}(H); \quad f \mapsto T_a f(x) = f(x/a)$$

is an isomorphism. Moreover it is clear that $\mathcal{D}_{(\mathfrak{M})}(H)$ is isomorphic to a subspace of $\mathcal{P}_{(\mathfrak{M})}(\mathbb{R}^n)$. Hence the conclusion since every subspace of a nuclear space is nuclear.

(b) Every Hausdorff countable inductive limit of nuclear spaces is nuclear.

(c) We know that the vector space $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ is dense in $\mathcal{E}_{(\mathfrak{M})}(\Omega)$. To conclude it suffices then to establish that $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ considered as a subspace of $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ is nuclear. This easily follows from (a).

Let $\|\cdot\|_{K,h,j}$ be any fundamental continuous semi-norm on $\mathcal{E}_{(\mathfrak{M})}(\Omega)$. We choose a compact subset H of Ω such that $K \subset H^\circ$ and $\psi \in \mathcal{D}_{(\mathfrak{M})}(H)$ identically 1 on a neighbourhood of K. Since $\|\cdot\|_{K,h,j}$ is a continuous semi-norm on $\mathcal{D}_{(\mathfrak{M})}(H)$, there is a fundamental continuous semi-norm $\|\cdot\|_{H,k,l}$ on $\mathcal{D}_{(\mathfrak{M})}(H)$ such that $\|\cdot\|_{K,h,j}$ is sub-nuclear with respect to $\|\cdot\|_{H,k,l}$, i.e. there are sequences $(c_n)_{n\in\mathbb{N}}$ of complex numbers and $(t_n)_{n\in\mathbb{N}}$ of $\mathcal{D}_{(\mathfrak{M})}(H)'$ such that

$$\|f\|_{K,h,j} \le \sum_{n=1}^{\infty} |c_n t_n(f)|\,, \quad \forall f \in \mathcal{D}_{(\mathfrak{M})}(H),$$

with

$$\sum_{n=1}^{\infty} |c_n| < \infty \quad \text{and} \quad |t_n(\cdot)| \le \|\cdot\|_{H,k,l}\,.$$

In particular, for every $f \in \mathcal{D}_{(\mathfrak{M})}(\Omega)$, we have

$$\|f\|_{K,h,j} = \|f\psi\|_{K,h,j} \le \sum_{n=1}^{\infty} |c_n t_n(f\psi)|$$

with

$$\sum_{n=1}^{\infty} |c_n| < \infty \qquad |t_n(\cdot\psi)| \le \|\cdot\psi\|_{H,k,l}\,.$$

As the multiplication map

$$M_\psi : \mathcal{D}_{(\mathfrak{M})}(H) \to \mathcal{D}_{(\mathfrak{M})}(H); \quad f \mapsto f\psi$$

is linear and continuous, there are $C > 0$ and a fundamental continuous semi-norm $\|\cdot\|_{H,k',l'}$ on $\mathcal{D}_{(\mathfrak{M})}(H)$ such that

$$\|M_\psi f\|_{H,k,l} \le C\,\|f\|_{H,k',l'}\,, \quad \forall f \in \mathcal{D}_{(\mathfrak{M})}(H).$$

This leads to

$$\|f\|_{K,h,j} \leq \sum_{n=1}^{\infty} \left| c_n {}^t M_\psi t_n(f) \right|, \quad \forall f \in \mathcal{D}_{(\mathfrak{M})}(\Omega),$$

with

$$\sum_{n=1}^{\infty} |c_n| < \infty \quad \text{and} \quad \left| {}^t M_\psi t_n(f) \right| \leq C \|f\|_{H,k',l'}, \quad \forall f \in \mathcal{D}_{(\mathfrak{M})}(\Omega).$$

This implies that $\mathcal{D}_{(\mathfrak{M})}(\Omega)$ considered as a subspace of $\mathcal{E}_{(\mathfrak{M})}(\Omega)$ is nuclear. Hence the conclusion.∎

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