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Composition followed by differentiation between weighted Bergman spaces and weighted Banach spaces of holomorphic functions

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Abstract

Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$ in the complex plane. Such a map induces through composition a linear composition operator $C_\phi : f \mapsto f \circ \phi$. We are interested in the combination of C_ϕ with the differentiation operator D, that is in the operator $DC_\phi : f \mapsto \phi' \cdot (f' \circ \phi)$ acting between weighted Bergman spaces and weighted Banach spaces of holomorphic functions.

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1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. For an analytic self-map ϕ of $\mathbb{D}$ the classical *composition operator* C_ϕ is given by

$$C_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f \circ \phi,$$

where $H(\mathbb{D})$ denotes the set of all analytic functions on $\mathbb{D}$. Combining this with differentiation we obtain the operator

$$DC_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \phi' \cdot (f' \circ \phi).$$

Composition operators occur naturally in various problems and therefore have been widely investigated. An overview of results in the classical setting of the Hardy spaces as well as an introduction to composition operators is given in the excellent monographs by Cowen and MacCluer (cf. [6]) and Shapiro (cf. [13]).
Next, let us explain the setting in which we are interested. Bounded and continuous functions $v : \mathbb{D} \to]0, \infty[$ are called *weights*. For such a weight v we define

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}.$$

Since, endowed with the weighted sup-norm $\|.\|_v$, this is a Banach space, we say that H_v^∞ is a *weighted Banach space of holomorphic functions*. These spaces arise naturally in several problems related to e.g. complex analysis, spectral theory, Fourier analysis, partial differential and convolution equations. Concrete examples may be found in [4]. Weighted Banach spaces of holomorphic functions have been studied deeply in [3], but also in [5] and [2].

The *weighted Bergman space* is defined to be the collection of all analytic functions $f \in H(\mathbb{D})$ such that

$$A_v^p := \{f \in H(\mathbb{D});\ \|f\|_{v,p} := \left(\int_{\mathbb{D}} |f(z)|^p v(z)\, dA(z)\right)^{\frac{1}{p}} < \infty\}, 1 \leq p < \infty,$$

where $dA(z)$ denotes the normalized area measure. The investigation of Bergman spaces has quite a long and rich history. An excellent introduction to Bergman spaces is given in [9]. In this article we characterize boundedness and compactness of operators $DC_\phi : A_{v,p} \to H_w^\infty$ in terms of the involved self-map ϕ and the weights v and w.

2 Basics

We study weighted spaces generated by the following class of weights. Let ν be a holomorphic function on $\mathbb{D}$ that does not vanish and is strictly positive on $[0,1[$. Moreover, we assume that $\lim_{r\to 1} \nu(r) = 0$. Then we define the weight v in the following way

$$v(z) := \nu(|z|^2) \text{ for every } z \in \mathbb{D}. \tag{2.1}$$

Examples include all the famous and popular weights, such as

1. the *standard weights* $v(z) = (1 - |z|^2)^\alpha$, $\alpha \geq 1$,
2. the *logarithmic weights* $v(z) = (1 - \log(1 - |z|^2))^\beta$, $\beta > 0$.
3. the *exponential weights* $v(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$, $\alpha \geq 1$.

For a fixed point $a \in \mathbb{D}$, we introduce a function

$$v_a(z) := \nu(\overline{a}z) \text{ for every } z \in \mathbb{D}.$$

Since ν is holomorphic on $\mathbb{D}$, so is the function v_a. Moreover, in particular, we will often assume that there is a constant $C > 0$ such that

$$\sup_{a\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{v(z)}{|v_a(z)|} \leq C. \tag{2.2}$$

Calculations show that the standard weights as well as the logarithmic weights satisfy condition (2.2), while the exponential weights do not fulfill condition (2.2).
Finally, we need some geometric data of the unit disk. A very important tool when dealing with operators such as defined above is the so-called *pseudohyperbolic metric* given by

$$\rho(z,a) := |\sigma_a(z)|,$$

where $\sigma_a(z) := \frac{a-z}{1-\overline{a}z}$, $z, a \in \mathbb{D}$, is the Möbius transformation which interchanges a and 0.

3 Results

Lemma 1 *Let $v(z) = h(|z|)$ for every $z \in \mathbb{D}$, where $h \in H(\mathbb{D})$ is a function whose Taylor series (at 0) has nonnegative coefficients. We assume additionally that v satisfies condition (2.2). Then there is a constant $C > 0$ such that, for all $f \in A_v^p$,*

$$|f(z)| \leq C^{\frac{1}{p}} \frac{\|f\|_{v,p}}{v(0)^{\frac{1}{p}}(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}}.$$

Proof. Recall that a weight v as defined above may be written as

$$v(z) := \max\{|g(\lambda z)|; |\lambda| = 1\} \text{ for every } z \in \mathbb{D}.$$

We will write $g_\lambda(z) := g(\lambda z)$ for every $z \in \mathbb{D}$. Next, fix $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Moreover, let $\alpha \in \mathbb{D}$ be an arbitrary point. We consider the map

$$T_{\alpha,\lambda} : A_v^p \to A_v^p,\ T_{\alpha,\lambda} f(z) = f(\sigma_\alpha(z))\sigma'_\alpha(z)^{\frac{2}{p}} g_\lambda(\sigma_\alpha(z))^{\frac{1}{p}}.$$

Then a change of variables yields

$$\begin{aligned}
\|T_{\alpha,\lambda} f\|_{v,p}^p &= \int_{\mathbb{D}} v(z)|f(\sigma_\alpha(z))|^p |\sigma'_\alpha(z)|^2 |g_\lambda(\sigma_\alpha(z))|\, dA(z) \\
&\leq \int_{\mathbb{D}} |f(\sigma_\alpha(z))| \frac{v(z)}{v(\sigma_\alpha(z))} |\sigma'_\alpha(z)|\, dA(z) \\
&\leq C \int_{\mathbb{D}} \int_{\mathbb{D}} |f(\sigma_\alpha(z))| v(\sigma_\alpha(z)) |\sigma'_\alpha(z)|^2\, dA(z) \\
&\leq C \int_{\mathbb{D}} v(t)|f(t)|^p\, dA(t) = C\|f\|_{v,p}^p.
\end{aligned}$$

Now put $h_\lambda(z) := T_{\alpha,\lambda}(z)$ for every $z \in \mathbb{D}$. By the mean value property we obtain

$$v(0)|h_\lambda(0)|^p \leq \int_{\mathbb{D}} v(z)|h_\lambda(z)|^p\, dA(z) = \|h_\lambda\|_{v,p}^p \leq C\|f\|_{v,p}^p.$$

Hence

$$v(0)|h_\lambda(0)|^p = v(0)|f(\alpha)|^p(1-|\alpha|^2)^2|g_\lambda(\alpha)| \leq C\|f\|_{v,p}^p.$$

Since λ was arbitrary we obtain that

$$v(0)|f(\alpha)|^p(1-|\alpha|^2)^2 v(\alpha) \leq C\|f\|_{v,p}^p$$

Thus,

$$|f(\alpha)| \leq C^{\frac{1}{p}} \frac{\|f\|_{v,p}}{v(0)^{\frac{1}{p}}(1-|\alpha|^2)^{\frac{2}{p}} v(\alpha)^{\frac{1}{p}}}.$$

Since α was arbitrary, the claim follows. □

Lemma 2 *Let $v(z) = h(|z|)$ for every $z \in \mathbb{D}$, where $h \in H(\mathbb{D})$ is a function whose Taylor series (at 0) has nonnegative coefficients. We assume additionally that v satisfies condition 2.2. Then for every $f \in A_v^p$ there is $C_v > 0$ such that*

$$|f(z) - f(w)| \leq C_v \|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}} v(w)^{\frac{1}{p}}}\right\} \rho(z,w)$$

for every $z, w \in \mathbb{D}$.

Proof. The proof is completely analogous to the proof given in [17]. Hence we omit it here. □

Lemma 3 *Let $v(z) = h(|z|)$ for every $z \in \mathbb{D}$, where $h \in H(\mathbb{D})$ is a function whose Taylor series (at 0) has nonnegative coefficients. We assume additionally that v satisfies condition (2.2). Then for $f \in A_v^p$ and $z \in \mathbb{D}$:*

$$|f'(z)| \leq \frac{M}{v(0)^{\frac{1}{p}}(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}\|f\|_{v,p}.$$

Proof. By Lemma 2 we have that

$$|f(z) - f(w)| \leq \frac{M}{v(0)^{\frac{1}{p}}}\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}}v(w)^{\frac{1}{p}}}\right\}\rho(z,w)\|f\|_{v,p}.$$

Now

$$\begin{aligned}
\left|\frac{f(z+h)-f(z)}{|h|}\right| &\leq \frac{M}{v(0)^{\frac{1}{p}}|h|}\max\left\{\frac{1}{(1-|z+h|^2)^{\frac{2}{p}}v(z+h)^{\frac{1}{p}}}, \frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\right\}\rho(z+h,z)\|f\|_{v,p} \\
&= \frac{M}{v(0)^{\frac{1}{p}}|h|}\max\left\{\frac{1}{(1-|z+h|^2)^{\frac{2}{p}}v(z+h)^{\frac{1}{p}}}, \frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\right\}\left|\frac{z+h-z}{1-\overline{(z+h)}z}\right|\|f\|_{v,p} \\
&= \frac{M}{v(0)^{\frac{1}{p}}}\max\left\{\frac{1}{(1-|z+h|^2)^{\frac{2}{p}}v(z+h)^{\frac{1}{p}}}, \frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\right\}\left|\frac{1}{1-\overline{(z+h)}z}\right|\|f\|_{v,p}.
\end{aligned}$$

Finally, let h tend to zero and obtain

$$|f'(z)| \leq \frac{M}{v(0)^{\frac{1}{p}}(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}\|f\|_{v,p}.$$

□

Proposition 4 *Let $v(z) = h(|z|)$ for every $z \in \mathbb{D}$, where $h \in H(\mathbb{D})$ is a function whose Taylor series (at 0) has nonnegative coefficients. We assume additionally that v satisfies condition (2.2). Then $DC_\phi : A_v^p \to H_w^\infty$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}\frac{w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{1+\frac{2}{p}}v(\phi(z))^{\frac{1}{p}}} < \infty. \tag{3.1}$$

Proof. First, we assume that (3.1) is satisfied. Applying Lemma 3 we obtain

$$\|DC_\phi f\|_w = \sup_{z\in\mathbb{D}} w(z)|\phi'(z)||f'(\phi(z))| \leq C\sup_{z\in\mathbb{D}}\frac{w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{1+\frac{2}{p}}v(\phi(z))^{\frac{1}{p}}}.$$

Hence $DC_\phi : A_v^p \to H_w^\infty$ must be bounded.
Conversely, let $a \in \mathbb{D}$ be arbitrary. Then there exists f_a^p in the unit ball of H_v^∞ such that $|f_a(a)|^p = \frac{1}{\tilde{v}(a)}$. Now put

$$g_a(z) := f_a(z)\varphi_a(z) \text{ for every } z \in \mathbb{D}.$$

Hence $\|g_a\|_{v,p}^p = \int_{\mathbb{D}} |g_a(z)|^p v(z)\, dA(z) \leq \sup_{z\in\mathbb{D}} v(z)|f_a(z)|^p \int_{\mathbb{D}} |\varphi_a(z)|^p\, dA(z) \leq K$. Moreover,

$$g_a'(z) = f_a'(z)\varphi_a(z) + f_a(z)\varphi_a'(z) \text{ for every } z \in \mathbb{D}.$$

Now, to show that the boundedness of the operator $DC_\phi : A_v^p \to H_w^\infty$ implies condition (3.1), we assume to the contrary that (3.1) is not satisfied. Thus, there is a sequence $(z_n)_n \subset \mathbb{D}$ such that

$$\frac{w(z_n)|\phi'(z_n)|}{(1-|\phi(z_n)|^2)^{1+\frac{2}{p}} v(\phi(z_n))^{\frac{1}{p}}} \geq n \text{ for every } n \in \mathbb{N}.$$

Next, we consider functions $g_n(z) := g_{\phi(z_n)}(z)$ for every $n \in \mathbb{N}$ as defined above. Obviously $(g_n)_n$ is contained in the closed unit ball of A_v^p and

$$c \geq w(z_n)|\phi'(z_n)||g_n'(\phi(z_n))| = \frac{w(z_n)|\phi'(z_n)|}{v(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{1+\frac{2}{p}}} \geq n$$

for every $n \in \mathbb{N}$ which is a contradiction. □

Next, we want to analyze under which conditions an operator $DC_\phi : A_v^p \to H_w^\infty$ is compact. To do this we need the following form of the Weak Compactness Theorem. The proof is similar to the case of composition operators acting on H^2. Thus we refer the reader to the monograph of Shapiro [13], Section 2.4.

Lemma 5 *Let v and w be arbitrary weights. A bounded operator $DC_\phi : A_v^p \to H_w^\infty$ is compact if and only if for every bounded sequence $(f_n)_n \subset A_v^p$ that converges to 0 uniformly on the compact subsets of $\mathbb{D}$, the sequence $(DC_\phi f_n)_n$ also converges to 0 in H_w^∞.*

Proposition 6 *Let $v(z) = h(|z|)$, $z \in \mathbb{D}$, where $h \in H(\mathbb{D})$ is a function whose Taylor series (at 0) has nonnegative coefficients. Moreover, we assume that v satisfies (2.2). Then the operator $DC_\phi : A_v^p \to H_w^\infty$ is compact if and only if*

$$\limsup_{|\phi(z)|\to 1} \frac{w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{1+\frac{2}{p}} v(\phi(z))^{\frac{1}{p}}} = 0.$$

Proof. Let $(f_n)_n$ be a bounded sequence in A_v^p that converges to zero uniformly on the compact subsets of $\mathbb{D}$. Let $M := \sup_n \|f_n\|_{v,p} < \infty$. Given $\varepsilon > 0$ there is $r > 0$ such that if $|\phi(z)| > r$, then

$$\frac{w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{1+\frac{2}{p}} v(\phi(z))^{\frac{1}{p}}} \leq \frac{\varepsilon}{2C_v}.$$

On the other hand, since $f_n \to 0$ uniformly on $\{u;\ |u| \leq r\}$, there is an $n_0 \in \mathbb{N}$ such that if $|\phi(z)| \leq r$ and $n \geq n_0$, then $w(z)|f_n'(\phi(z))||\phi'(z)| < \frac{\varepsilon}{2}$. Now, an application of Lemma 3 yields

$$\begin{aligned} \sup_{z\in\mathbb{D}} w(z)|f_n'(\phi(z))||\phi'(z))| &\leq \sup_{|\phi(z)|\leq r} w(z)|f_n'(\phi(z))||\phi'(z)| + \sup_{|\phi(z)|>r} w(z)|f_n'(\phi(z))||\phi'(z)| \\ &\leq \frac{\varepsilon}{2} + \sup_{|\phi(z)|>r} \frac{C_v w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}+1} v(\phi(z))^{\frac{1}{p}}} < \varepsilon. \end{aligned}$$

Thus, the claim follows.
Conversely, we suppose that $DC_\phi : A_{v,p} \to H_w^\infty$ is compact and that there are $\delta > 0$ and $(z_n)_n \subset \mathbb{D}$ with $|\phi(z_n)| \to 1$ such that

$$\frac{w(z_n)|\phi'(z_n)|}{(1-|\phi(z_n)|^2)^{1+\frac{2}{p}} v(\phi(z_n))^{\frac{1}{p}}} \geq \delta.$$

Since $|\phi(z_n)| \to 1$ there exist natural numbers $\alpha(n)$ with $\lim_{n\to\infty} \alpha(n) = \infty$ such that $|\phi(z_n)|^{\alpha(n)} \geq \frac{1}{2}$ for every $n \in \mathbb{N}$.
Next, for every $n \in \mathbb{N}$ we consider the function

$$g_n(z) := f_n(z)\sigma_{\phi(z_n)}^{1+\frac{2}{p}}(z) z^{\alpha(n)},$$

where $f_n^p \in H_v^\infty$ such that $\|f_n^p\|_v \leq 1$ and $|f_n(\phi(z_n))|^p = \frac{1}{\tilde{v}(\phi(z_n))}$. Thus, $(g_n)_n \subset A_v^p$ converges to 0 uniformly on the compact subsets of $\mathbb{D}$ and we obtain

$$\begin{aligned} \|DC_\phi f_n\|_w &\geq w(z_n)|\phi'(z_n)||f_n'(\phi(z_n))| \\ &\geq \frac{w(z_n)|\phi'(z_n)||\phi(z_n)|^{\alpha(n)}}{\tilde{v}(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{1+\frac{2}{p}}} \\ &\geq \frac{1}{2}\frac{w(z_n)|\phi'(z_n)|}{\tilde{v}(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{1+\frac{2}{p}}} \geq \frac{1}{2}\delta. \end{aligned}$$

Applying Lemma 5 we arrive at a contradiction and the claim follows. □

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