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THE MOST PROBABLE CONFIGURATIONS FOR A QUADRATIC GRAVITY MODEL

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Abstract. After the Wheeler-DeWitt equation is solved perturbatively for the quantum cosmological wavefunction, it can be maximized to determine the most probable configurations. This method also distinguishes between the different types of boundary conditions.

1. Introduction

Infinities which arise in the perturbative quantization of the Einstein-Hilbert action and supergravity are known to be absent from superstring theories in ten dimensions. This includes the elimination of large-order divergences that occur in field theories and bosonic string theories which can be removed after the introduction of supersymmetry, BRST symmetry and modular invariance.

The ten-dimensional heterotic string theory, which is anomaly-free at one loop and predicts phenomenologically viable gauge groups through a reduction to four dimensions, gives rise to a four-dimensional effective action containing only scalar and metric fields, after setting all other fields equal to zero. This model represents a modification of the action for general relativity and a scalar field. In addition to the Ricci scalar and the kinetic term for the dilaton field, there is a quadratic term with a coupling to a scalar field [1], and the theory has nonsingular classical solutions which would ameliorate the regularity of the quantum cosmological model. It has been demonstrated that, amongst the classical solutions of the four-dimensional model, there are metrics that approximately describe the $K = 0$ Friedmann-Robertson-Walker space-time [4]. This space-time is cosmological background with exponential expansion and it admits supersymmetry [7]. At first order in an expansion in powers of a sigma-model parameter, the heterotic string effective action is a quadratic gravity theory, which is unitary because the curvature combination is a Gauss-Bonnet term, dynamically nontrivial as a a result of the coupling of the scalar factor through a factor $\frac{e^{-\Phi}}{g_4^2}$ and has improved renormalizability properties [5].

Renormalizability in the generalized sense has been established only the coupling of scalar fields to curvature terms of linear and quadratic

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order [10]. These results are not extended to quartic curvature terms which are considered in with the heterotic string effective action. Nevertheless, the heterotic string theory has known finiteness properties, and the equivalence of the quartic sector with a higher-derivative super-Yang-Mills theory [9] confirms the relevance of the quantum quantum cosmological wavefunction with the inclusion of quartic curvature terms. Nevertheless, during the inflationary epoch, the reduction in the magnitude of the higher-order curvature terms extends to quartic combinations, and the overall weighting of the quartic sector would be determined by the balance between the number and coefficients of such terms and the reduced magnitude.

The wavefunctions represent solutions to a Wheeler-DeWitt equation, which, being the operator equivalent of a classical equation, would yield the probability distribution of an ensemble of metric and matter fields Therefore, an analogy with kinetic equations in many-body problems is evident. It would represent a smooth, macroscopic version of a stochastic quantization of gravity [12]. The extrema of the probability distributions, based on the no-boundary and tunneling wavefunctions and corrections derived for the quadratic model, are determined. The normalization of these wavefunctions depends on the the finiteness of the L^2 norm. When it is finite, the wavefunction may be divided by the normalization factor. If the L^2 norm is infinite, conditional probabilities may be used. Nevertheless, the maximization of $|\Psi(a,V)|^2$ indicates the most probable configurations.

2. **Second-Order Wheeler-DeWitt Equation**

The quadratic action with a potential term equals

(2.1)
$$I = \int d^4x\sqrt{-g}\left[\frac{1}{\kappa^2}R+\frac{1}{2}(D\Phi)^2+\frac{e^{-\Phi}}{4g_4^2}(R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}-4R_{\mu\nu}R^{\mu\nu}+R^2)-2V(\Phi)\right].$$

Restriction to the minisuperspace of Friedmann-Robertson-Walker metrics

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \tag{2.2}$$

the one-dimensional action after the addition of a boundary term is

$$I = \int \left[6a(-\dot{a}^2 + K) + \frac{1}{2}a^3\dot{\Phi}^2 + 2\frac{e^{-\Phi}}{g_4^2}\dot{a}(\dot{a}^2 + 3K) - 2a^3V(\Phi)\right]. \tag{2.3}$$

Canonical quantization of this action begins with the momenta

$$P_a = -12a\dot{a} + 6\frac{e^{-\Phi}}{g_4^2}\dot{\Phi}(\dot{a}^2 + K) \tag{2.4}$$

$$P_\Phi = a^3\dot{\Phi} + 2\frac{e^{-\Phi}}{g_4^2}\dot{a}(\dot{a}^2 + 3K).$$

The Wheeler-DeWitt equation to first order in $\frac{e^{-\Phi}}{g_4^2}$ is

(2.5)

$$
\begin{aligned}
H\Psi &= \left(H_0 + \frac{e^{-\Phi}}{g_4^2} H_1\right)\Psi = 0 \\
H_0 &= \frac{1}{24a}\frac{\partial}{\partial a}\frac{1}{a}\frac{\partial}{\partial a} - \frac{1}{2a^3}\frac{\partial^2}{\partial \Phi^2} - 6aK + 2a^3 V(\Phi) \\
H_1 &= \frac{1}{a^4}\left(\frac{K}{4} - \frac{1}{576a^4}\right)\frac{\partial}{\partial a} + \frac{1}{576a^7}\frac{\partial^2}{\partial a^2} - \frac{1}{1728a^6}\frac{\partial^3}{\partial a^3} \\
&\quad + \frac{1}{a^5}\left(\frac{7K}{4} - \frac{35}{576a^4}\right)\frac{\partial}{\partial \Phi} + \frac{1}{2a^8}\frac{\partial^2}{\partial a \partial \Phi} - \frac{1}{64a^7}\frac{\partial^3}{\partial a^2 \partial \Phi^2} + \frac{1}{864a^6}\frac{\partial^4}{\partial a^3 \partial \Phi}.
\end{aligned}
$$

If $|V|$ is not significantly less than one, the semiclassical approximation is not valid and higher orders of quantum gravity become important. The quadratic gravity action is renormalizable in the generalized sense and can be viewed as sufficient for the inclusion of higher-order curvature terms. To first order in $\frac{e^{-\Phi}}{g_4^2}$, this equation is fourth-order if the Φ derivatives are included and third-order when the Φ derivatives are not included.

For a slow-roll potential, $\left|V^{-1}\frac{dV}{d\Phi}\right| \ll 1$ and the second-order Wheeler-DeWitt equation, derived from the differential operator H_0, becomes an Airy equation

$$
\frac{d^2\psi}{dz^2} + z\psi = 0 \tag{2.6}
$$

$$
z = -K\left(\frac{18}{V}\right)^{\frac{2}{3}}\left(1 - \frac{a^2 V}{3K}\right) \qquad K \neq 0
$$

with basis solutions $Ai(-z)$ and $Bi(-z)$. If $K = 0$, a and V do not have to be rescaled and z may be chosen to be $(4V)^{-\frac{2}{3}}a^2 V = 4^{-\frac{2}{3}}a^2 V^{\frac{1}{3}}$ [4]

To first order in $\frac{e^{-\Phi}}{g_4^2}$,

$$
\left(H_0 + \frac{e^{-\Phi}}{g_4^2}H_1\right)\left(\Psi_0 + \frac{e^{-\Phi}}{g_4^2}\Psi_1\right) \approx 0 \tag{2.7}
$$

and $H_0\left(\frac{e^{-\Phi}}{g_4^2}\Psi_1\right) \approx -\frac{e^{-\Phi}}{g_4^2}H_1\Psi_0$. The derivatives of Ψ can be regarded as negligible in the slow-roll approximation, where the potential is approximately flat near the initial vacuum. Let

$$
H_0 = H_0^{slow-roll} - \frac{1}{2a^3}\frac{\partial^2}{\partial \Phi^2} \tag{2.8}
$$

and

(2.9)

$$\left(H_0^{slow-roll} - \frac{1}{2a^3}\frac{\partial^2}{\partial\Phi^2}\right)\left(\frac{e^{-\Phi}}{g_4^2}\Psi_1\right) = \frac{e^{-\Phi}}{g_4^2}H_0^{slow-roll}\Psi_1 - \frac{1}{2a^3}\frac{e^{-\Phi}}{g_4^2}\Psi_1 + \frac{1}{a^3}\frac{e^{-\Phi}}{g_4^2}\frac{\partial\Psi_1}{\partial\Phi} - \frac{1}{2a^3}\frac{e^{-\Phi}}{g_4^2}\frac{\partial^2\Psi_1}{\partial\Phi^2}.$$

When $\frac{\partial^2\Psi_0}{\partial\Phi^2}$ is kept in $H_0\Psi_0$, the solution is given by the product of a Bessel function with an index dependeing on ν and $e^{i\nu\Phi}$, and it is customary to choose the ground state mode with $\nu = 0$. After setting the derivatives $\frac{\partial\Psi_1}{\partial\Phi}$ and $\frac{\partial^2\Psi_1}{\partial\Phi^2}$ equal to zero, the term $\frac{1}{2a^3}\frac{e^{-\Phi}}{g_4^2}\Psi_1$ is not large for $\frac{e^{-\Phi}}{g_4^2} < 1$, $a \gg 1$. During the inflationary epoch, it can be neglected in comparison with $6aK\Psi_1$.

3. **No-Boundary Wavefunction**

The no-boundary wavefuncion [11][13] is

$$\Psi_{0NB} = \frac{Ai(-z)}{Ai(-z_0)} = \frac{Ai\left(K\left(\frac{18}{V}\right)^{\frac{2}{3}}\left(1 - \frac{a^2V}{3K}\right)\right)}{Ai\left(K\left(\frac{18}{V}\right)^{\frac{2}{3}}\right)} \quad (3.1)$$

when $K = 1$ or -1. While the no-boundary wavefunction is defined by a path integral over compact four-manifolds, corresponding to the conventional choice $K = 1$, the other values of K are possible if the range of coordinates in the flat or hyperbolic sections is finite.

Even if the classical solution for $a(t)$ has a lower bound greater than zero, the wavefunction still can take values at $a = 0$, since the minisuperspace $\{a, \Phi\}$ includes $a = 0$. If $K = 0$, and z is set equal to $2^{-\frac{2}{3}}a^2(2V)^{\frac{1}{3}}$, this variable would vanish at $a = 0$, and a normalization factor such as $Ai(-z_c)$, $z_c = z(a_c)$, where $a_c > 0$ can be used.

To analyze these wavefunctions, the following asymptotic formulae are useful:

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}}z^{-\frac{1}{4}}e^{-\frac{2}{3}z^{\frac{3}{2}}} \qquad z \to \infty \quad (3.2)$$

$$Ai(-z) \sim \frac{1}{\sqrt{\pi}}z^{-\frac{1}{4}}sin\left[\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right].$$

In the asymptotic expansion of $Ai(-z_0)$ as $z_0 = -K\left(\frac{18}{V}\right)^{\frac{2}{3}} \to -\infty$ for $K = 1$, the normalization factor tends to $2\sqrt{\pi}(-z_0)^{\frac{1}{4}}e^{\pm\frac{2}{3}(-z_0)^{\frac{3}{2}}}$ if $K = 1$. Thus the exponential prefactor is $e^{\pm\frac{2}{3}(-z_0)^{\frac{3}{2}}} = e^{\pm 12\frac{K^{\frac{3}{2}}}{V}}$. The negative sign favours inflation, whereas the positive sign favours $V = 0$. When $K = 0$, the change in the sign across the $V(\Phi) = 0$ boundary leads to different asymptotics for the wavefunction. If the positive sign for the exponent in the prefactor is $e^{\pm\frac{2}{3}(-z_c)^{\frac{3}{2}}=e^{\pm\frac{1}{3}a_c^3|2V|^{\frac{1}{2}}}}$ [5].

As $(-z) \to \infty$, and the conventional sign is chosen is chosen in the exponential,

$$\Psi_{0NB}(z) \to \left(\frac{a^2V}{3K} - 1\right)^{-\frac{1}{4}} e^{-i\cdot\frac{12}{V}\left(\frac{a^2V}{3K}-1\right)^{\frac{3}{2}}+\frac{i\pi}{4}} e^K$$

$12\overline{\ \ V.(3.3)}$The probability distribution then would be

$$p(a,V) = |G|^{\frac{1}{2}}(a)\,|\Psi_{0NB}|^2 = |G|^{\frac{1}{2}}(a)\left(\frac{a^2V}{3K} - 1\right)^{-\frac{1}{2}} e^{K^{\frac{3}{2}}\frac{24}{V}}.(3.4)$$

where G_{AB} is the minisuperspace metric [8]. Since $|G|^{\frac{1}{2}}$ depends only on a in the minisuperspace of Friedmann-Robertson-Walker metrics, it will not affect the vanishing of the derivative of the probability with respect to V, and the extrema would remain unchanged.

For an unnormalized distribution, $p(a,V)$ is maximized at $V = 0$. Probabilities are not larger than 1, and a cut-off is required. If $(\Psi^*\Psi)_{max}(a,V) = 1$ when

$$2a^2V = 6K\left(1 + e^{\frac{48K^{\frac{3}{2}}}{V}}\right) \tag{3.5}$$

then

$$e^{\frac{48}{V}} = e^{\frac{a^2}{16}}\frac{1}{1+e^{\frac{48}{V}}}. \tag{3.6}$$

Let $w = e^{\frac{48}{V}}$ such that

$$(1+w)ln\ w = \frac{a^2}{16} \tag{3.7}$$

and

$$V = 48\left\{ln\left[\frac{\frac{a^2}{16}}{ln\left(\frac{a^2}{16}\right)} - 1\right]\right\}^{-1} \qquad for\ a^2 \gg 16. \tag{3.8}$$

For $\frac{a^2}{16} \ll 1$, $w \simeq 1$. If $w = 1+\epsilon$,

$$ln\ w = ln(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \ \ ... = \frac{a^2}{16}\cdot\frac{1}{2+\epsilon} = \frac{a^2}{32} \tag{3.9}$$

and

$$\frac{48}{V} \simeq \frac{a^2}{32}. \tag{3.10}$$

From both Eqs. (3.8) and (3.10), $a^2V > 3K$ within the range of validity for the classically allowed region.

Consider the vanishing of $\frac{dp}{dV}$ given by

$$4a^2V(-48K^{\frac{3}{2}}) + 576K^{\frac{5}{2}} - 4a^2V^2 = 0 \tag{3.11}$$

which has the solution

$$(3.12)\qquad V=\frac{1}{4}\left[-96K^{\frac{3}{2}}\pm\frac{1}{a^2}\cdot 96K^{\frac{3}{2}}a^2\left[1+\frac{2304}{9216}\frac{K^{\frac{5}{2}}}{K^3a^4}\right]^{\frac{1}{2}}\right].$$

The positive root $\frac{3K}{a^4}-\frac{3}{16}\frac{K^{\frac{1}{2}}}{a^8}+...$ is not located within the classically allowed region.

The wavefunction in the classical forbidden region [4] [13]

$$(3.13)\qquad \Psi_{0NB}=\frac{1}{2}\left[1-\frac{a^2V}{3K}\right]^{-\frac{1}{4}}exp\left[\frac{12K^{\frac{3}{2}}}{V}\left(1-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right]$$

yields the probability distribution

(3.14)

$$p(a,V)=\frac{1}{4}|G|^{\frac{1}{2}}\left[1-\frac{a^2V}{3K}\right]^{-\frac{1}{2}}exp\left[\frac{24K^{\frac{3}{2}}}{V}\left(1-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right].$$

Normalization of this probability distribution follows from the integral

(3.15)

$$\int\int p(a,V)dadV=\frac{1}{4}\int\int|G|^{\frac{1}{2}}\left[1-\frac{a^2V}{3K}\right]^{-\frac{1}{2}}exp\left[\frac{24K^{\frac{3}{2}}}{V}\left(1-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right]dadV$$

Letting $V\to 0$,

(3.16)

$$\frac{24K^{\frac{3}{2}}}{V}\left[1-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\right]=\frac{24K^{\frac{3}{2}}}{V}\left[\frac{3}{2}\frac{a^2V}{3K}-\frac{3}{8}\left(\frac{a^2V}{3K}\right)^2+...\right]$$
$$=12a^2K^{\frac{1}{2}}-\frac{a^4V}{K^{\frac{1}{2}}}+...\,.$$

Since $\frac{a^2V}{3K}<1$, $\frac{a^4V}{K^{\frac{1}{2}}}<3a^2K^{\frac{1}{2}}$ and $12a^2K^{\frac{1}{2}}-\frac{a^4V}{K^{\frac{1}{2}}}+...>9a^2K^{\frac{1}{2}}$. Finiteness of the integral (3.15) in the neighbourhood of $V=0$ occurs only if there is an upper bound for the scale factor determined by the inflationary interval or a normalization factor of the form $4|G|^{-\frac{1}{2}}e^{-Ba^2K^{\frac{1}{2}}}$ for some $B>9$. This factor can be included if the conditional probability distribution $\frac{p(a,V)}{p(a|V\to 0)}$ is used. It follows from Eq.(3.16) that $p(a|V\to 0)\approx\frac{1}{4}|G|^{\frac{1}{2}}e^{Ba^2K^{\frac{1}{2}}}$.

Then
(3.17)

$$\int\int \frac{p(a,V)}{p(a|V\to 0)} dadV$$
$$= \int\int \frac{\frac{1}{4}|G|^{\frac{1}{2}}\left(1-\frac{a^2V}{3K}\right)^{-\frac{1}{2}} exp\left[\frac{24K^{\frac{3}{2}}}{V}\left(1-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right]}{\frac{1}{4}|G|^{\frac{1}{2}}e^{Ba^2K^{\frac{1}{2}}} da\ dV}$$
$$= \int\int e^{-Ba^2K^{\frac{1}{2}}}\left(1-\frac{a^2V}{3K}\right)^{-\frac{1}{2}} exp\left[\frac{24K^{\frac{3}{2}}}{V}\left(1-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right] da\ dV$$

and there is no divergence in the limit $V\to 0$ and $a\to\infty$.

Setting $B=12$, a normalization of the wavefunction would be given by
(3.18)
$$\frac{1}{N}\int\int da\,dV\,e^{-12a^2K^{\frac{1}{2}}}\left[1-\frac{a^2V}{3K}\right]^{-\frac{1}{2}} exp\left[\frac{24K^{\frac{3}{2}}}{V}\left(1-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right]=1.$$

Let $\tilde{y}=\left(1-\frac{a^2V}{3K}\right)^{\frac{1}{2}}$ such that $V=\frac{3K}{a^2}(1-\tilde{y}^2)$. Then the condition is

$$(3.19)\qquad -\frac{6K}{N}\int\int da\ d\tilde{y}\frac{e^{-12a^2K^{\frac{1}{2}}}}{a^2} exp\left[8a^2K^{\frac{1}{2}}\frac{\tilde{y}^2+\tilde{y}+1}{\tilde{y}+1}\right]=1.$$

When V is very negative, $\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\gg 1$, and the increasing exponential would be unbounded as $V\to\infty$. If the integration range for V is chosen to be $\left[-V_0,\frac{3K}{a^2}\right]$, the integrand will be regular there is a minimum value a_0 for the scale factor a. The normalization would be given by

$$(3.20)\qquad \frac{6K}{N}\int_{a_0}^{\infty}\int_0^{y_0}\frac{e^{-12a^2K^{\frac{1}{2}}}}{a^2} exp\left[8a^2K^{\frac{1}{2}}\frac{\tilde{y}^2+\tilde{y}+1}{\tilde{y}+1}\right]=1.$$

Since N is a constant, it would not affect the extremum of the conditional probability distribution with respect to V.

Let $\frac{a^2V}{3K}=y<1$. Then the integral equals

$$(3.21)\qquad \frac{1}{4}(1-y)^{-\frac{1}{2}}\ e^{-Ba^2K^{\frac{1}{2}}}\int dV\ exp\left[\frac{36K^{\frac{3}{2}}}{y}-9K^{\frac{3}{2}y^2+\ldots}V\right]$$

An upper bound for the potential V_{max} is necessary for finiteness because
$exp\left[\frac{36K^{\frac{3}{2}}y-9K^{\frac{3}{2}}y^2+\ldots}{V}\right]\to 1$ as $V\to\infty$. With a coefficient dependent

only on a, the condition of the vanishing of the derivative with respect to V is unchanged. Then

(3.22)

$$\frac{dp}{dV} = |G|^{\frac{1}{2}} \frac{a^2}{24K} \left[1 - \frac{a^2V}{3K}\right]^{-\frac{3}{2}} exp\left[\frac{24K^{\frac{3}{2}}}{V}\left(1 - \left(1 - \frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right]$$
$$+ \frac{1}{2}\left[1 - \frac{a^2V}{3K}\right]^{-\frac{1}{2}} \left\{ \left(-\frac{12K^{\frac{3}{2}}}{V^2}\right)\left(1 - \left(1 - \frac{a^2V}{3K}\right)^{\frac{3}{2}} + \frac{48K^{\frac{3}{2}}}{V}\left(1 - \frac{a^2V}{3K}\right)^{\frac{1}{2}}\left(\frac{a^2}{3K}\right)\right\}$$
$$\cdot exp\left[\frac{24K^{\frac{3}{2}}}{V}\left(1 - \left(1 - \frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right]$$

which equals zero if

(3.23)

$$\left[\frac{a^2V}{6K} - \left(1 - \frac{a^2V}{3K}\right)\left(\frac{24K^{\frac{3}{2}}}{V}\right)\right]^2 = \left(1 - \frac{a^2V}{3K}\right)^3\left(1 + \frac{a^2V}{6K}\right)\left(\frac{24K^{\frac{3}{2}}}{V}\right)^2.$$

Then $y = 0$ or
(3.24)

$$y^4 + y^3 - 5y^2 + \left(\left(\frac{V}{24K^{\frac{3}{2}}}\right)^2 + 4\left(\frac{V}{24K^{\frac{3}{2}}}\right) + 3\right)y - 4\left(\frac{V}{24K^{\frac{3}{2}}}\right) = 0.$$

Then

$$(3.25)\quad \left(\frac{a^2V}{3K}\right)^4 + \left(\frac{a^2V}{3K}\right)^3 - 5\left(\frac{a^2V}{3K}\right)^2$$
$$+ \left(\left(\frac{V}{24K^{\frac{3}{2}}}\right)^2 + \frac{V}{6K^{\frac{3}{2}}} + 3\right)\left(\frac{a^2V}{3K}\right) - \frac{V}{6K^{\frac{3}{2}}} = 0$$

Dividing by V,

(3.26)

$$8V^3 + 4\left[\frac{6K}{a^2} + \frac{1}{(48)^2K^3}\left(\frac{6K}{a^2}\right)^3\right]V^2 + 2\left[-5\left(\frac{6K}{a^2}\right)^2 + \frac{1}{12K^{\frac{3}{2}}}\left(\frac{6K}{a^2}\right)^3\right]V$$
$$+ \left[3 - \frac{1}{2a^2K^{\frac{1}{2}}}\right]\left(\frac{6K}{a^2}\right)^3$$
$$= 8V^3 + 4\left[\frac{6K}{a^2} + \frac{3}{32a^6}\right]V^2 - \left[\frac{180K^2}{a^4} - \frac{36K^{3\frac{3}{2}}}{a^6}\right]V + \left[3 - \frac{1}{2a^2K^{\frac{1}{2}}}\right]\left(\frac{6K}{a^2}\right)^3 = 0.$$

Given the parameters for the cubic equation $(2V)^3+\alpha(2V)^2+\beta(2V)+\gamma=0$,

(3.27)

$$\begin{aligned} p_{\alpha,\beta} &= -\frac{\alpha^2}{3}+\beta = -\frac{1}{3}\left[\frac{6K}{a^2}+\frac{3}{32a^6}\right]^2 - \left[\frac{180K^2}{a^4}-\frac{18K^{\frac{3}{2}}}{a^6}\right] \\ &= -\frac{192}{a^4}+\frac{18K^{\frac{3}{2}}}{a^6}-\frac{3K}{16a^8}-\frac{3}{1024a^{12}} \\ q_{\alpha,\beta} &= 2\left(\frac{\alpha}{3}\right)^3-\frac{\alpha\beta}{3}+\gamma \\ &= \frac{1}{a^6}\left(304K^3-\frac{72K^{\frac{5}{2}}}{a^2}-\frac{17K^2}{4a^4}+\frac{9K^{\frac{3}{2}}}{16a^6}+\frac{3K}{512a^8}+\frac{1}{16384a^{12}}\right) \end{aligned}$$

For the $K=0$ Friedmann-Robertson-Walker space-time,

$$\begin{aligned} Q(K=0) &= \left[\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2\right]_{K=0} \\ &= \frac{1}{12}\left(-\frac{1}{1024a^8}\right)^3+\frac{1}{a^{12}}\left(\frac{1}{32768a^{12}}\right)^2 \\ &= -\frac{1}{2^{30}a^{36}}+\frac{1}{2^{30}a^{36}}=0 \end{aligned} \tag{3.28}$$

The solutions to the cubic equation when $K=0$ are

$$\begin{aligned} V^{(1)} &= -\frac{3}{64a^6} \\ V^{(2)} &= V^{(3)}=0 \end{aligned} \tag{3.29}$$

Since the signature is $(+---)$, it is consistent with that of the space-time background in conventional inflationary models with a grand unified potential. There, the exponential expansion occurs in a deSitter phase with a positive cosmological constant resulting from a positive extremum of the potential.

In a supersymmetric string theory, a negative cosmological term can be derived initially from the Kähler potential, and, in the $K=0$ Friedmann-Robertson-Walker space-time, the scale factor is exponential. For example, the effective scalar potential

$$\begin{aligned} V_F &= \frac{1}{8(Re\ T)^3}\left\{\frac{1}{3}|2(Re\ T)W'-3W|^2\right\} \\ W &= W_0+Ae^{-aT} \end{aligned} \tag{3.30}$$

yields a vacuum with a negative cosmological term [3]. The form of the potential between this minimum and the axis of vanishing V is comparable with that required by maximization of the probability distribution.

Since the signature is $(+---)$, it is consistent with that of the space-time background in conventional inflationary models with a grand unified potential. Therefore, the exponential expansion occurs in a deSitter phase with a positive cosmological constant resulting from a positive extremum of the potential. The inflationary potential in grand unified theories can be derived from a superpotential

$$V_B(\Phi) = \frac{e^{-3\sigma_0}}{16} g_4^2 e^{\Phi} |W(S)|^2 \left[1 - 2S\frac{W'(S)}{W(S)}\right]^2 \left[-1 + 2S\frac{W'(S)^2 - W(S)W''(S)}{W(S)^2}\right]^{-1}$$
$$W(S) = c + h\left(\frac{3S}{b_0}\right) e^{-\frac{3S}{2b_0}} \tag{3.31}$$

with $S = \frac{e^{-\Phi}}{g_4^2}$, where Φ is the dilaton field, g_4 is the four-dimensional string coupling and b_0 defined for the gauge group after setting the modulus field equal to zero [2]. The parameters c and h can be chosen such that $\left|\frac{V_B'(\Phi)}{V_B(\Phi)}\right| \ll 1$ and $\left|\frac{V_B''(\Phi)}{V_B(\Phi)}\right| \ll 1$ for sufficiently large Φ and inflation occurs. While the antisymmetric tensor field and the gaugino condensate are known to introduce terms in the potential which can break supersymmetry, the minimum value of the potential remains zero, and it is lifted in higher-order perturbation theory at non-zero temperature [2]. At the extremum, supersymmetry is not preserved. The time for the decrease of the effective potential to zero can be maximized with respect to the expectation values of the gauge singlet fields.

A more general form of the potential is necessary. Given that the probability distribution is maximized for a rapid decrease of a negative potential, it would represent the end of an interval of expansion of the universe generated by the scale factor of a metric admitting superymmetry-generating spinors [4]. The expression for V in Eq.(3.30) with a negative extremum does not have the form of a slow-roll potential required for an extended inflationary epoch. Instead, the standard positive potential derived for grand unified theories is necessary. It would follow that the interval of inflation passes through a supersymmetric phase to an exponential expansion without supersymmetry.

For arbitrary real values of z,

$$p(a, V) = |G|^{\frac{1}{2}} \frac{|Ai(-z)|^2}{|Ai(-z_0)|^2}. \tag{3.32}$$

From the derivative

$$\frac{dz}{dV} = \frac{2K}{3}(18)^{\frac{2}{3}} V^{-\frac{5}{3}} \left(1 + \frac{a^2 V}{6K}\right) \tag{3.33}$$

it follows that
(3.34)

$$\frac{dp}{dV} = |G|^{\frac{1}{2}} \frac{2|Ai(-z)|\left[|Ai(-z_0)||Ai'(-z)|\left(-\frac{dz}{dV}\right) - |Ai(-z)||Ai'(-z_0)|\left(-\frac{dz_0}{dV}\right)\right]}{|Ai(-z_0)|^3}$$

when any discontinuity in the derivative would be discarded. It vanishes when
(3.35)

$$|Ai(-z_0)Ai'(-z)|\left(-\frac{2K}{3}\right)(18)^{\frac{2}{3}}V^{-\frac{5}{3}}\left(1+\frac{a^2V}{6K}\right) - |Ai(-z)Ai'(-z_0)|\left(-\frac{2K}{3}\right)\cdot(18)^{\frac{2}{3}}V^{-\frac{5}{3}} = 0.$$

For positive values of the Airy function and its derivative,

$$Ai(-z) = exp\left[\frac{Ai'(-z_0)}{Ai(-z_0)}\int \frac{d(-z)}{1+\frac{a^2V}{6K}}\right] \tag{3.36}$$

Since

$$\begin{aligned}\int \frac{d(-z)}{\left(1+\frac{a^2V}{6K}\right)} &= \int \frac{d(-z)}{-\frac{3}{2K}(18)^{-\frac{2}{3}}V^{\frac{5}{3}}\frac{d(-z)}{dV}} \\ &= -\frac{2K}{3}(18)^{\frac{2}{3}}\int V^{-\frac{5}{3}}dV \\ &= K\left(\frac{18}{V}\right)^{\frac{2}{3}} = -z_0.\end{aligned} \tag{3.37}$$

and

$$Ai(-z) = exp\left[-z_0\frac{Ai'(-z_0)}{Ai(-z_0)}\right]. \tag{3.38}$$

The solution to the third-order differential equation resulting from the inclusion of the derivatives in the scale factor in H_1, and the vanishing of derivatives with respect to the scalar field, has an additional term containing the Airy functions
(3.39)

$$\begin{aligned}\Psi = &\left(1+C_1\frac{e^{-\Phi}}{g_4^2}\right)Ai(-z) + C_2Bi(-z) \\ &+ C_3\frac{e^{-\Phi}}{g_4^2}\frac{72\pi}{Ai\left(K\left(\frac{18}{V}\right)^{\frac{2}{3}}\right)}Bi(-z)\cdot\int\frac{da}{a^3}\left[-KAi'(-z)+\frac{1}{36}\left(\frac{V}{18}\right)^{\frac{2}{3}}Ai'''(-z)\right]Ai(-z) \\ &- C_4\frac{e^{-\Phi}}{g_4^2}\frac{72\pi}{Ai\left(K\left(\frac{18}{V}\right)^{\frac{2}{3}}\right)}Ai(-z)\cdot\int\frac{da}{a^3}\left[-KAi'(-z)+\frac{1}{36}\left(\frac{V}{18}\right)^{\frac{2}{3}}Ai'''(-z)\right]Bi(-z).\end{aligned}$$

Since $p(a,V) = N(a)\Psi^*(a,V)\Psi(a,V)$,

$$\frac{dp}{dV} = N(a)\left[\frac{d\Psi^*(a,V)}{dV}\Psi(a,V) + \Psi^*(a,V)\frac{d\Psi(a,V)}{dV}\right]. \tag{3.40}$$

Setting $C_2 = C_3 = 0$ for regularity, and substituting the expression (3.1), the derivative $\frac{d\Psi}{dV}$ equals

$$\begin{aligned}
&\left(1 + C_1\frac{e^{-\Phi}}{g_4^2}\right)\frac{d}{dV}Ai(-z) \\
&- 2C_1\frac{e^{-\Phi}}{g_4^2}V'(\Phi)^{-1}Ai(-z) \\
&- 72\pi C_4\frac{e^{-\Phi}}{g_4^2}e^{-i\frac{\pi}{4}}\left(\frac{a^2V}{3K} - 1\right)^{-\frac{1}{4}}e^{-\frac{12K^{\frac{3}{2}}}{V}} \\
&\int\frac{da}{a^3}\left[-Kf_1Ai'(-z) + \frac{1}{18}2^{\frac{2}{3}}Vf_3Ai'''(-z)\right]Bi(-z) \\
&+ 2C_4\frac{e^{-\Phi}}{g_4^2}V'(\Phi)^{-1}\frac{72\pi}{Ai\left(K\left(\frac{18}{V}\right)^{\frac{2}{3}}\right)}Ai(-z) \\
&\cdot\int\frac{da}{a^3}\left[-KAi'(-z) + \frac{1}{36}\left(\frac{V}{18}\right)^{\frac{2}{3}}Ai'''(-z)\right]Bi(-z)
\end{aligned} \tag{3.41}$$

where f_1 and f_3 are conversion factors for the derivatives with respect to the definition of z with the potential V replaced by $72K^{\frac{3}{2}}V$. It may be noted that, for $a \gg 1$, $z \to \frac{1}{(4V)^{\frac{2}{3}}}(24a^2K^{\frac{1}{2}}V) = \frac{12K^{\frac{1}{2}}}{2^{\frac{1}{3}}}a^2V^{\frac{1}{3}}$. The integral over a yields a function that decreases rapidly with respect to a, and an overall factor of $(\frac{a^2V}{3K} - 1)^{-\frac{1}{2}}e^{-\frac{24K^{\frac{3}{2}}}{V}}$ arises. The extremum will not be shifted far from the value derived previously.

The solutions to the field equations correspond to most probable configurations. For $K = 0$ metrics,

$$\Psi_{0NB} = \frac{Ai(2^{-\frac{1}{3}}a^2V^{\frac{1}{3}})}{Ai(2^{-\frac{1}{3}}a_c^2V^{\frac{1}{3}})}. \tag{3.42}$$

Then

$$\Psi_{0NB}\underset{V\gg 1}{\longrightarrow}\left(\frac{a_c}{a}\right)^{\frac{1}{2}}e^{-\frac{\sqrt{2}}{3}(a^3-a_c^3)V^{\frac{1}{2}}} \tag{3.43}$$

and there is a maximum of $Ai(-z)$ at ϵ^* such that

$$\begin{aligned}
-2^{-\frac{1}{3}}a^2V^{\frac{1}{3}} &= \epsilon^* \\
V &= -2\frac{\epsilon^{*3}}{a^6}.
\end{aligned} \tag{3.44}$$

Suppose that potential has the form [7]

(3.45)

$$a(t) = a_0 e^{\lambda t}$$

$$\Phi(t) \simeq \sqrt{C}(t - t_0) + D$$

$$V(\Phi) \sim e^{-\frac{3\sigma_0}{16}} k_1^2 + \left(2k_1 k_2 - \frac{9k_1^2}{b_0^2}\right)\frac{e^{-\Phi}}{g_4^2} \;+\; \left(k_2^2 - \frac{243}{24b_0^2}k_1^2 - \frac{5k_1k_2}{b_0}\right)\frac{e^{-2\Phi}}{g_4^4} \;+\; \dots .$$

The equality between the leading terms in the potential occurs when

$$-2\frac{\epsilon^{*3}}{a^6} = k_2 \frac{e^{-2\Phi}}{g_4^4} \tag{3.46}$$

or

$$-2\epsilon^{*3} a_0^{-6} e^{-6\lambda t} \approx \frac{k_2}{g_4^4} e^{-2\sqrt{C}t} e^{-2D} \tag{3.47}$$

$$\lambda = \frac{\sqrt{C}}{3}$$

$$-2\epsilon^{*3} a_0^{-6} = \frac{k_2}{g_4^4} e^{-2D}.$$

4. **The Tunneling Wavefunction**

The tunneling wavefunction, defined by the condition $i\frac{\partial\Psi}{partiala} > 0$, is given by a different combination of the basis of Airy functions [13]

$$\Psi_T = \frac{Ai(-z) + iBi(-z)}{Ai(-z_0) + iBi(-z_0)}. \tag{4.1}$$

In the classically allowed range,

$$|\Psi_{0T}^{all.}|^2 = \frac{1}{\pi^2}\left[\frac{a^2 V}{3K} - 1\right]^{-\frac{1}{2}} e^{-\frac{24K^{\frac{3}{2}}}{V}} \tag{4.2}$$

such that the derivative with respect to the potential vanishes when

$$-\frac{a^2}{12K} + \left(\frac{a^2V}{3K} - 1\right)\left(\frac{24K^{\frac{3}{2}}}{V}\right) = 0. \tag{4.3}$$

The roots of the equation

$$4V^2 - 192K^{\frac{3}{2}}V + 576\frac{K^{\frac{5}{2}}}{a^2} = 0 \tag{4.4}$$

are

In the classically forbidden region [14],

$$\Psi_{0T}^{for.} = \left(1 - \frac{a^2V}{3K}\right)^{-\frac{1}{4}} exp\left(-\frac{12K^{\frac{3}{2}}}{V}\left(1 - \left(1 - \frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right) \tag{4.5}$$

and the probability distribution is
(4.6)
$$p(a,V)=(-G)^{\frac{1}{2}}\left(1-\frac{a^2V}{3K}\right)^{-\frac{1}{2}}exp\left(-\frac{24K^{\frac{3}{2}}}{V}\left(1-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)\right).$$

With a negative sign in the exponential, as $V\to 0$,
(4.7)
$$-\frac{24K^{\frac{3}{2}}}{V}\left(1-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\right)=-12a^2K^{\frac{1}{2}}+\frac{a^4V}{2K^{\frac{1}{2}}}-...<-9a^2K^{\frac{1}{2}}$$

and the integral would converge for $a\in[0,\infty]$ and $V\in[0,V_{max}]$ without a conditional probability distribution because

$$\text{(4.8)}\qquad \int_0^{\infty}dV(1-y)^{-\frac{1}{2}}exp\left[-\frac{48K^{\frac{3}{2}}}{V}\left(\frac{3}{2}y-\frac{3}{8}y^2+...\right)\right]<\infty$$

when V_{max} is bounded.

The derivative vanishes when
(4.9)
$$\frac{a^2V}{6K}+\left(1-\frac{a^2V}{3K}\right)\left(\frac{24K^{\frac{3}{2}}}{V}\right)=-\left(1-\frac{a^2V}{3K}\right)^{\frac{3}{2}}\left(\frac{24K^{\frac{3}{2}}}{V}\right)\left(1+\frac{a^2V}{3K}\right)$$

or
(4.10)
$$8V^3+4\left[\frac{6K}{a^2}+\frac{3}{32a^6}\right]V^2-2\left[\frac{180K^2}{a^4}+\frac{18K^{\frac{3}{2}}}{a^6}\right]V+\left[3+\frac{1}{2a^2K^{\frac{3}{2}}}\right]\left(\frac{6K}{a^2}\right)^3=0$$

From the parameters

$$\text{(4.11)}\quad p_{\alpha\beta}=-\frac{192K^2}{a^4}-\frac{18K^{\frac{3}{2}}}{a^6}-\frac{3K}{16a^8}-\frac{3}{1024a^{12}}$$
$$q_{\alpha\beta}=\frac{304K^3}{a^6}+\frac{72K^{\frac{5}{2}}}{a^8}-\frac{17K^2}{4a^{10}}-\frac{9K^{\frac{3}{2}}}{16a^{12}}+\frac{3K}{512a^{14}}+\frac{1}{16384a^{18}}$$

it may be deduced that $Q=0$ for $K=0$ Friedmann-Robertson-Walker space-times. Again, the solutions for the potential are given by Eq.(3.26).

5. A Solution by Matrix Methods

Another form of the Wheeler-DeWitt equation can be derived by matrix methods. By Eq.(2.4),

$$\text{(5.1)}\qquad (P_a+12a\dot{a})\dot{a}=6\frac{e^{-\Phi}}{g_4^2}\dot{\Phi}\dot{a}\left(\dot{a}^2+K\right)$$
$$(P_\Phi-a^3\dot{\Phi})\dot{\Phi}=2\frac{e^{-\Phi}}{g_4^2}\dot{\Phi}\dot{a}(\dot{a}^2+3K).$$

Then

$$(P_a - 12a\dot{a})\dot{a} - 3(P_\Phi - a^3\dot{\Phi})\dot{\Phi} = -12K\frac{e^{-\Phi}}{g_4^2}\dot{\Phi}\dot{a}. \tag{5.2}$$

which has the form

$$\begin{pmatrix}\dot{a} & \dot{\Phi}\end{pmatrix}\begin{pmatrix} -12a & 6K\frac{e^{-\Phi}}{g_4^2} \\ 6K\frac{e^{-\Phi}}{g_4^2} & 3a^3 \end{pmatrix}\begin{pmatrix}\dot{a} \\ \\ \dot{\Phi}\end{pmatrix} = (-P_a \ 3P_\Phi)\begin{pmatrix}\dot{a} \\ \\ \dot{\Phi}\end{pmatrix}. \tag{5.3}$$

The linear equation has the solution

$$\begin{pmatrix}\dot{a} \\ \\ \dot{\Phi}\end{pmatrix} = \frac{1}{12\left(a^4 + K^2\frac{e^{-2\Phi}}{g_4^4}\right)}\begin{pmatrix} -a^3 & 2K\frac{e^{-\Phi}}{g_4^2} \\ 2K\frac{e^{-\Phi}}{g_4^2} & 4a \end{pmatrix}\begin{pmatrix}-P_a \\ \\ 3P_\Phi\end{pmatrix} \tag{5.4}$$

and

$$\begin{aligned} \dot{a} &= \frac{1}{12\left(a^4 + K^2\frac{e^{-2\Phi}}{g_4^2}\right)}\left(a^3P_a + 6K\frac{e^{-\Phi}}{g_4^2}P_\Phi\right) \\ \dot{\Phi} &= \frac{1}{6\left(a^4 + K^2\frac{e^{-2\Phi}}{g_4^4}\right)}\left(-K\frac{e^{-\Phi}}{g_4^2}P_a + 6aP_\Phi\right). \end{aligned} \tag{5.5}$$

The Hamiltonian is

$$H = -6\left(a - \frac{e^{-\Phi}}{g_4^2}\dot{\Phi}\dot{a}\right)\left(\dot{a}^2 + K\right) + \frac{1}{2}a^3\dot{\Phi}^2 + 2a^3V(\Phi). \tag{5.6}$$

Substituting the expressions for $\dot{a}$ and $\dot{\Phi}$ in the Hamiltonian gives

(5.7)

$$\begin{aligned} H = -6\Bigg[a - \frac{e^{-\Phi}}{g_4^2}\frac{1}{6\left(a^4 + K^2\frac{e^{-2\Phi}}{g_4^4}\right)}\left(-K\frac{e^{-\Phi}}{g_4^2}P_a + 6aP_\Phi\right) \\ \cdot \frac{1}{12\left(a^4 + K^2\frac{e^{-2\Phi}}{g_4^4}\right)}\left(a^3P_a + 6K\frac{e^{-\Phi}}{g_4^2}P_\Phi\right)\Bigg] \\ \left[\left(\frac{1}{12\left(a^4 + K^2\frac{e^{-2\Phi}}{g_4^4}\right)}\left(a^3P_a + 6K\frac{e^{-\Phi}}{g_4^2}P_\Phi\right)\right)^2 + K\right] \\ + \frac{1}{2}a^3\left[\frac{1}{6\left(a^4 + K^2\frac{e^{-2\Phi}}{g_4^4}\right)}\left(-K\frac{e^{-\Phi}}{g_4^2}P_a + 6aP_\Phi\right)\right]^2 + 2a^3V(\Phi). \end{aligned}$$

In the limit $\frac{e^{-\Phi}}{g_4^2} \to 0$,

(5.8)

$$lim_{\frac{e^{-\Phi}}{g_4^2}\to 0} H = H_0 = -6a \cdot \left(\frac{1}{12a^4}a^3 P_a + K\right)^2 + \frac{1}{2}a^3\left(\frac{1}{a^6}P_\Phi^2\right) + 2a^3 V(\Phi)$$
$$= -\frac{1}{24}\frac{1}{a}P_a\frac{1}{a}P_a - 6aK + \frac{1}{2a^3}P_\Phi^2 + 2a^3 V(\Phi).$$

With $P_a \to -i\frac{\partial}{\partial a}$ and $P_\Phi \to -\frac{\partial}{\partial\Phi}$, the standard form of the leadinng order differential operator in the Wheeler-DeWitt equation is recovered for gravity coupled to a scalar field.

$$(5.9)\quad H_0\Psi = \frac{1}{24a^2}\frac{\partial^2\Psi}{\partial a^2} - \frac{1}{24a^3}\frac{\partial}{\partial a}\frac{\partial\Psi}{\partial a} - 6aK\Psi - \frac{1}{2a^3}\frac{\partial^2\Psi}{\partial\Phi^2} + 2a^3V(\Phi)\Psi.$$

The $O\left(\frac{e^{-\Phi}}{g_4^2}\right)$ part of the Hamiltonian is

$$(5.10)\quad \frac{e^{-\Phi}}{g_4^2}H_1' = 6\frac{e^{-\Phi}}{g_4^2}\frac{1}{6a^4}\cdot 6aP_\Phi\frac{1}{12a^4}a^3P_a\left(\left(\frac{1}{12a}P_a\right)^2 + K\right)$$
$$- 6a\frac{1}{12a^4}\left[a^3P_a\frac{1}{12a^4}6K\frac{e^{-\Phi}}{g_4^2} + 6K\frac{e^{-\Phi}}{g_4^2}P_\Phi\frac{1}{a}P_a\right]$$
$$+ \frac{1}{2}a^3\frac{1}{6a^4}\left(-6K\frac{e^{-\Phi}}{g_4^2}P_a\frac{1}{6a^3}P_\Phi - \frac{6K}{a^3}P_\Phi\left(\frac{e^{-\Phi}}{g_4^2}P_a\right)\right)$$

and the differential operator is

(5.11)

$$H_1' = -\frac{1}{2a^4}\frac{\partial^2}{\partial a\partial\Phi}\left(-\frac{1}{144}\frac{1}{a}\frac{\partial}{\partial a}\frac{1}{a}\frac{\partial}{\partial a} + K\right) + \frac{K}{4}\left(\frac{2}{a^4}\frac{\partial^2}{\partial a\partial\Phi} - \frac{4}{a^5}\frac{\partial}{\partial\Phi}\right)$$
$$+ \frac{K}{2a}\left(\frac{2}{a^3}\frac{\partial^2}{\partial a\partial\Phi} - \frac{1}{a^3}\frac{\partial}{\partial a} - \frac{3}{a^4}\frac{\partial}{\partial\Phi}\right)$$
$$= \frac{1}{288a^4}\frac{\partial^2}{\partial a\partial\Phi}\left(\frac{1}{a}\frac{\partial}{\partial a}\frac{1}{a}\frac{\partial}{\partial a}\right) + K\left(\frac{1}{a^4}\frac{\partial^2}{\partial a\partial\Phi} - \frac{1}{2a^4}\frac{\partial}{\partial a} - \frac{5}{2a^5}\frac{\partial}{\partial\Phi}\right)$$
$$= \frac{1}{288a^6}\frac{\partial^4}{\partial a^3\partial\Phi} - \frac{1}{288a^7}\frac{\partial^3}{\partial a^2\partial\Phi} + K\left(\frac{1}{a^4}\frac{\partial^2}{\partial a\partial\Phi} - \frac{1}{2a^4}\frac{\partial}{\partial a} - \frac{5}{2a^5}\frac{\partial}{\partial\Phi}\right)$$

Again, the next term in the expansion of Ψ can be solved iteratively through the equation

$$(5.12)\qquad H_0\left(\frac{e^{-\Phi}}{g_4^2}\Psi_1\right) = -\frac{e^{-\Phi}}{g_4^2}H_1'\Psi_0.$$

The extrema of Ψ_0 with the two boundary conditions have been given in §3 and §4. These would be shifted infinitesimally by the addition of $\frac{e^{-\Phi}}{g_4^2}\Psi_1$.

6. **The Most Probable Value of the Hypersurface Curvature**

A cosmological background solving the equations for the scale factor and the dilaton has been derived for the Lagrangian containing only the first derivative of $a(t)$ with the inclusion of the heterotic string potential. The metric is that of the $K = 0$ Friedmann-Robertson-Walker space-time, and consistent supersymmetric physical theories can be formulated on this expanding geometry [7]. The condition of ah path integral over four-manifolds with compact three-sections would require the $K = 1$ model in the minisuperspace of Friedmann-Robertson-Walker metrics. The viability of these two cosmological metrics is indicative of the possible three-geometries that could be included in the path integral over four-manifolds. The restriction to compact manifolds without boundaries would be confirmed by higher probability for the $K = 1$ metrics. This could be followed by a matching with the $K = 0$ Friedmann-Robertson-Walker space-time at a boundary between the Planck scale and the inflationary epoch.

The no-boundary condition yields a probability distribution that is extremized by a negative potential, and a metric admitting supersymmetry-generating spinors is a solution to the gravitational field equations. Since supersymmetry is considered to present in the fundamental quantum theory of the universe at the earliest times, this result is indicative of the choice of compact four-manifolds from the Friedmann-Robertson-Walker with $K = 1,\ 0, -1$.

A Mellin transform of the closed-form differential equation for the wavefunction, where $\Psi^*(s,a) = \int_0^\infty w^{s-1}\Psi(w,a)dw$, produces a difference differential equation [14] for $\Psi^*(s,a)$

(6.1)

$$
\begin{aligned}
&L_1(s,a)E_1^2\Psi^*(s,a) + L_2(s,a)E_1\Psi^*(s,a) + L_3(s,a)\Psi^*(s,a) = 0\\
&L_1(s,a) = -K(s+1)(s+2)D_2^2\\
&\qquad\qquad + \frac{1}{4a^6}[s^6 + 11s^5 + 50s^4 + 125s^3 + 205s^2 + 242s + 152]\\
&L_2(s,a) = a^2\frac{g_4^2}{6}(s+1)(aD_2^3 + 6D_2^2) + \frac{3}{2}g_4^2(a(s+1)D_2 + (s+1)^3)\\
&\qquad\qquad + ag_4^2(s+1)^2(s+2)D_2\\
&L_3(s,a) = a^4g_4^4[4a^3D_2 + a^4D_2^2]\\
&E_1\Psi^*(s,a) = \Psi^*(s+1,a).
\end{aligned}
$$

After specifying the first two values $\Psi*(0,a)$ and $\Psi*(1,a)$, the solution to this two-variable recursion relation for large s and the inverse transform yields

(6.2)

$$\Psi(w,a) \approx \frac{1}{2\pi i}\int_{\gamma-iN_0}^{\gamma+iN_0} \Psi^*(s,a)w^{-s}ds$$
$$+ \frac{1}{2\pi i}\int_{\gamma+iN_0}^{\gamma+i\infty} (-1)^s \frac{(4a^6)^{s-2}}{\Gamma(s-2)^3}\left[ag_4^2 D_2 + \frac{3}{2}g_4^2\right]^{s-2} w^{-s}ds \cdot \left[KD_2^2 - \frac{38}{a^6}\right]^{-1}$$
$$\left\{a\frac{g_4^2}{6}(aD_2^3 + 6D_2^2)\Psi^*(1,a) + \frac{3}{2}g_4^2\left(\frac{5}{3}aD_2 + 1\right)\Psi^*(1,a)\right.$$
$$\left. + a^4 g_4^4(4a^3 D_2 + a^4 D_2^2)\Psi^*(0,a)\right\}$$
$$+ \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+iN_0} (-1)^s \frac{(4a^6)^{s-2}}{\Gamma(s-2)^3}\left[ag_4^2 D_2 + \frac{3}{2}g_4^2\right]^{s-2} w^{-s}ds \cdot \left[KD_2^2 - \frac{38}{a^6}\right]^{-1}$$
$$\left\{a\frac{g_4^2}{6}(aD_2^3 + 6D_2^2)\Psi^*(1,a) + \frac{3}{2}g_4^2\left(\frac{5}{3}aD_2 + 1\right)\Psi^*(1,a)\right.$$
$$\left. + a^4 g_4^4(4a^3 D_2 + a^4 D_2^2)\Psi^*(0,a)\right\}$$

where $D_2\Psi^*(s,a) = \frac{d}{da}\Psi^*(s,a)$ and integration is defined over an imaginary line with real part γ, which can have an infinitesimal value to be located away from any potential singularities of the integrand, although it may be set equal to zero because $\frac{1}{\Gamma(s)}$ is nonsingular in the complex s plane.

At the origin of the expansion in this quantum cosmological model,

(6.3)

$$lim_{a\to 0}\Psi(w,a) = \frac{1}{2\pi i}lim_{a\to 0}\int_{\gamma-iN_0} \gamma + iN_0\Psi^*(s,a)w^{-s}$$
$$- \frac{1}{2\pi i}lim_{a\to 0}\int_{\gamma-iN_0}^{\gamma+iN_0} (-1)^s \frac{(4a^6)^{s-2}}{\Gamma(s-2)^3}\left[ag_4^2 D_2 + \frac{3}{2}g_4^2\right]^{s-2} w^{-s}ds$$
$$\cdot \left[KD_2^2 - \frac{38}{a^6}\right]^{-1}\left\{g_4^2\left(\frac{a}{6}(aD_2^3 + 6D_2^2) + \frac{3}{2}\left(\frac{5}{3}aD_2 + 1\right)\right)\Psi^*(1,a)\right.$$
$$\left. + a^4 g_4^4(4a^3 D_2 + a^4 D_2^2)\Psi^*(0,a)\right\}.$$

Since $lim_{a\to 0}\left[KD_2^2 - \frac{36}{a^6}\right]^{-1} \to \frac{a^6}{38}$ when $\Psi^*(0,a)$ and $\Psi^*(1,a)$ are regular in a, the second integral vanishes in this limit [6]. However, it is possible still to distinguish between $K = 1, 0, -1$ three-sections for $a \neq 0$. The functions
$g_4^2\left(\frac{a}{6}(aD_2^3 + 6D_2^2) + \frac{3}{2}\left(\frac{5}{3}aD_2 + 1\right)\right)\Psi^*(1,a)$ and $g_4^4 a^4(4a^3 D_2 + a^4 D_2^2)\Psi^*(0,a)$

can be expanded in an orthonormal basis of eigenfunctions of the operator $KD_2^2 - \frac{38}{a^6}$, corresponding to the eigenvalues λ_{1n} and λ_{0n}. In the region near the maximum values of these two functions D_2^2 will be negative, implying that $KD_2^2 - \frac{38}{a^6}$ is negative for $K = 1$ and the approximated by a truncated series consisting of eigenfunctions with negative eigenvalues. It follows that the second term in Eq.(6.2) would have the same sign as the first if the other factors in the integral combine constructively with integral of $\frac{1}{2\pi i}\Psi^*(s,a)$ for $|s| \leq N_0$ and $a > 0$ before the beginning of the inflationary era.

For $s = iy$, $-N_0 \leq y \leq N_0$,

(6.4)

$$(-1)^s a^{6s} w^{-s} = \left(-\frac{a^6}{w}\right)^{iy}$$
$$= e^{-\pi y} e^{iy\ ln\left(\frac{a^6}{w}\right)} = e^{-\pi y}\left[cos\ \left(y\ ln\left(\frac{a^6}{w}\right)\right) + i\ sin\ \left(y\ ln\left(\frac{a^6}{w}\right)\right)\right]$$

whereas

$$\Gamma(iy-2)^3 = \frac{1}{(iy-2)^3(iy-1)^3(iy)^3}\Gamma(iy+1)^3 \tag{6.5}$$

implying that the remaining terms have the form

(6.6)

$$\frac{i}{2\pi^2}\int_{-i\gamma-N_0}^{-i\gamma+N_0} dy e^{-\pi y}\frac{y^3(y+i)^3(y+2i)^3}{\Gamma(iy+1)^3}$$
$$\left[cos\left(y\ ln\left(\frac{a^6}{w}\right)\right) + i\ sin\left(y\ ln\left(\frac{a^6}{w}\right)\right)\right]$$
$$\left[ag_4^2 D_2 + \frac{3}{2}g_4^2\right]^{iy-2}\left[\sum_{n\in I_{max,1}} \lambda_{1n}^{-1} c_{1n}\psi_{1n}(a) + \sum_{n\in I_{max,0}} \lambda_{0n}^{-1} c_{0n}\psi_{0n}(a)\right]$$

where $I_{max,0}$ and $I_{max,1}$ are the index sets representing the major contribution of the series expansion of the two functions near the maxima. An evaluation of real and imaginary parts of the expression (6.6) would be sufficient to determine the change in the magnitude from the sum of the first and second integrals. If the summation is constructive, compact four-manifolds at the Planck time, compatible with the no-boundary condition, are favoured.

7. **Conclusion**

The quadratic gravity model was used to determine the most probable configuration through extremization with respect to the potential. For large values of the scale factor, without the time derivative of the scalar field approximately zero, a constant value for the potential was found. The extremum was shown to be shifted infinitesimally with the inclusion of the next-order term in the Hamiltonian. This result would be altered by the introduction of the time-dependence of the scalar field, which is consistent with the final states of inflationary potentials. The maxima of the probability distributions, defined by the vanishing of $\frac{dp}{dV}$ and $\frac{d^2p}{dV^2} < 0$, represent viable physical configurations if the values of V at the critical points are compatible with inflationary expansion. The description of cosmology with the no-boundary condition is supported by the existence of a wavefunction that is regular in the limit $a \to 0$ and consistent in the inflationary epoch. The wavefunctions are defined for a class of metrics for which the scale factor $a(t)$ is not specified. A comparison can be made between the extrema and solutions to the fields equations, including those found for the K=0 Friedmann-Robertson-Walker metrics. Within this class of metrics, the potential was required to decrease rapidly as a function of the scale factor, and a swift end to the first phase of the inflationary epoch would be predicted. It is likely that such a solution could be relevant only towards the end of the interval of inflation. More generally, the maxima of the probability distribution may be used to determine the cosmological parameters for these configurations.

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