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Weighted Banach spaces of holomorphic functions on the upper half-plane

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Abstract

In 2001 Bierstedt [1] asked if the weighted space of holomorphic functions $Hv_0(G)$ on the upper half-plane must have the approximation property under the conditions of Holtmanns [5]. Under these conditions she had shown that $Hv_0(G)''$ and $Hv(G)$ are isometrically isomorphic. The problem remains open in general, but in the present paper we give a positive answer for weights with two additional conditions. Actually we can even show the existence of a basis.

1. Introduction

In 1993 Bierstedt, Bonet und Galbis [2] investigated weighted spaces $Hv_0(G)$ of holomorphic functions for radial weights on balanced domains $G \subset \mathbb{C}^N, N \geq 1$. They showed that $Hv_0(G)$ has the bounded approximation property and that the polynomials are dense whenever they are contained in $Hv_0(G)$. For starshaped domains and admissible weights Kaballo and Vogt [6] had already proved the approximation property by a different method. More recently Stanev [11] studied weighted spaces of holomorphic functions on the upper half-plane. He gave a characterization when the spaces are not trivial, and with one of his examples one can construct a weighted space of holomorphic functions with an unbounded weight which has the bounded approximation property; see Example 23 below. In her thesis Holtmanns [5] investigated biduals of weighted spaces of holomorphic functions on the upper half-plane. She presented conditions on the weight v such that $Hv_0(G)''$ and $Hv(G)$ are isometrically isomorphic.

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2. Notation, main result

Throughout this article, we use the following notation. Let G be the upper half-plane, $G := \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$ and, $v : G \to \mathbb{R}_+$ a weight on G, i.e. a continuous function which satisfies the following conditions:

(i) $v > 0$ on G,

(ii) $\lim_{\operatorname{Im} z \to 0} v(z) = 0$,

(iii) there exists $0 < r_0 < 1$ with $v(z) \leq v(z + ir)$ for all $z \in G$ and $0 < r \leq r_0$,

(iv) for each $\varepsilon > 0$ there exists $b = b(\varepsilon) > 0$ such that $v(z) \geq b$ for all $z \in G$ with $\operatorname{Im} z \geq \varepsilon$,

(v) v is bounded.

The first three conditions were introduced by Holtmanns [5]. She did not require conditions (iv) and (v) for her work, but these conditions seem to be neceassary for the result given below. Define

$$Hv(G) := \{f \in H(G);\ ||f||_v := \sup_{z \in G} v(z)|f(z)| < \infty\},$$

$$Hv_0(G) := \{f \in H(G);\ vf \text{ vanishes at infinity on } G\}.$$

$Hv_0(G)$ is a closed subspace of $Hv(G)$, and both spaces are complete, hence Banach spaces, where $Hv_0(G)$ carries the induced norm.

The following is the main result of this article.

Theorem 1. *Let G be the upper half-plane and v a weight on G which satisfies conditions (i)-(v). Then $Hv_0(G)$ has a basis.*

Section 3 below is devoted to some preparations. The proof of Theorem 1 follows in section 4 and uses a result of Lusky [9].

3. Preparations

First we have to define some properties of sequences of linear operators.

Definitions 2. Let X be a given Banach space. For a fixed p with $1 \leq p \leq \infty$ we say that a sequence of continuous linear operators $V_n : X \to X$ *factors uniformly through l_p^m's* with respect to λ if there are suitable integers $m_n \in \mathbb{N}$ and continuous linear operators

$$T_n : X \to l_p^{m_n},\ S_n : l_p^{m_n} \to X,$$

with

$$V_n = S_n T_n,\ \sup_n ||T_n|| \leq \lambda \text{ and } \sup_n ||S_n|| \leq \lambda.$$

A sequence of bounded linear operators $V_n : X \to X$ of finite rank is called *commuting approximating sequence* (c.a.s.) if $\lim_{n\to\infty} V_n x = x$ for all $x \in X$ and $V_n V_m = V_{\min(n,m)}$

whenever $n \neq m$. If there exists such a sequence $(V_n)_{n\in\mathbb{N}}$, then X is said to have the *commuting bounded approximation property* (CBAP). If $V_nV_m = V_{\min(n,m)}$ holds, in addition, even for $n = m$ then X is said to have a *finite dimensional Schauder decomposition* (FDD). It is known that there are Banach spaces with CBAP which do not have FDD.

In 1996 Lusky [9] presented the following result which we will use in the case $p = \infty$ to show that $Hv_0(G)$ has a basis.

Theorem 3. *(Lusky) Let X have a commuting approximating sequence $(V_n)_{n\in\mathbb{N}}$ such that $V_n - V_{n-1}$ factors uniformly through l_p^m's for some $1 \leq p \leq \infty$. Then X has a basis.*

With the theorem above our problem is reduced to showing that $Hv_0(G)$ has a commuting approximating sequence $\{V_n\}_{n=1}^{\infty}$ such that $V_n - V_{n-1}$ factors uniformly through l_∞^m's. In the sequel some technical tools are given which are needed for the proof. In her thesis [5] Holtmanns defined linear operators Θ_n as follows:

Definition 4. (Holtmanns) For $f \in Hv_0(G)$ let

$$\Theta_n : Hv_0(G) \to Hv_0(G),\ n \in \mathbb{N},\ \Theta_n f := f_n$$

$$\text{with } f_n(z) := f(z + \frac{i}{n})\sqrt[n]{\frac{1}{z+i}} \text{ for } z \in G.$$

The main branch of the n-th root is well-defined since $z \to \frac{1}{z+i}$ maps G into the set $\{z \in \mathbb{C}\ ;\ \mathrm{Im} z < 0 \text{ and } |z| < 1\}$. The functions f_n are holomorphic on G since $z + i \neq 0$ for all $z \in G$.

Lemma 5. *(Holtmanns) Θ_n is well-defined and continuous as an operator from $Hv_0(G)$ into $Hv_0(G)$. $\Theta_n f$ converges to f in the compact-open topology, $f \in Hv_0(G)$.*

Lemma 6. *Let $f \in Hv_0(G)$ and Θ_n be as defined before. For each $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and a compact set $K \subset G$ with $v(z)|\Theta_n f(z) - f(z)| \leq \varepsilon$ for all $z \in G \setminus K$ and for any fixed $n \in \mathbb{N}, n \geq n_0$.*

Proof: Let $\varepsilon > 0$ be given. Set $\tilde{\varepsilon} = \frac{1}{4}\varepsilon$. $f \in Hv_0(G)$ means that there exist $L > 0$ and $0 < l < \frac{1}{2}$ with

$$v(z)|f(z)| \leq \tilde{\varepsilon}\ \forall\ z \in G \setminus [-L, L] \times i[l, L].$$

Set $K := [-L, L] \times i[\frac{l}{2}, L]$. For all $z \in G \setminus K$ the following inequality holds for $n \in \mathbb{N}$ large enough such that condition (iii) can be applied:

$$\begin{aligned} v(z)|\Theta_n f(z) - f(z)| &\leq v(z)(|f_n(z) - f(z+\tfrac{i}{n})| + |f(z+\tfrac{i}{n}) - f(z)|) \\ &\leq v(z)|f(z+\tfrac{i}{n})\sqrt[n]{\tfrac{1}{z+i}} - f(z+\tfrac{i}{n})| \\ &\quad + v(z)|f(z+\tfrac{i}{n})| + v(z)|f(z)| \\ &\leq v(z+\tfrac{i}{n})|f(z+\tfrac{i}{n})||\sqrt[n]{\tfrac{1}{z+i}} - 1| \\ &\quad + v(z+\tfrac{i}{n})|f(z+\tfrac{i}{n})| + v(z)|f(z)|. \end{aligned}$$

Let us now show that $v(z+\frac{i}{n})|f(z+\frac{i}{n})| \leq \tilde{\varepsilon}$ for $n \in \mathbb{N}$ large enough. Two cases are possible:
Case 1: $|\mathrm{Re}z| > L$ or $\mathrm{Im}z > L$. Then $z \notin K \Rightarrow z+\frac{i}{n} \notin K \Rightarrow v(z+\frac{i}{n})|f(z+\frac{i}{n})| \leq \tilde{\varepsilon}$.
Case 2: $\mathrm{Im}z < \frac{l}{2}$ and $|\mathrm{Re}z| \leq L$. Then there exists $n_0 \in \mathbb{N}$ with $\frac{1}{n} < \frac{l}{2}$ for all $n \in \mathbb{N}, n \geq n_0$. $z+\frac{i}{n} = x+i(y+\frac{1}{n})$ with $y+\frac{1}{n} < \frac{l}{2}+\frac{1}{n} \leq \frac{l}{2}+\frac{l}{2} = l$
$\Rightarrow \mathrm{Im}(z+\frac{i}{n}) < l \Rightarrow v(z+\frac{i}{n})|f(z+\frac{i}{n})| \leq \tilde{\varepsilon}$.
On the other hand, $\sup_{z\in G}|\sqrt[n]{\frac{1}{z+i}}| = \sup_{z\in G}\sqrt[n]{\frac{1}{|z+i|}} = 1$ $\forall n \in \mathbb{N}$ since $|z+i| \geq |\mathrm{Im}z|+1 \geq 1$ $\forall z \in G$, and hence $|1-\sqrt[n]{\frac{1}{z+i}}| \leq 2$.

Using these two estimates in the right hand side of the above inequality yields

$$v(z)|\Theta_n f(z) - f(z)| \leq 2\tilde{\varepsilon}+\tilde{\varepsilon}+\tilde{\varepsilon} \leq \varepsilon$$

for each $z \in G \setminus K$.

Corollary 7. *With Lemmas 5 and 6 it follows that for $f \in Hv_0(G)$ and for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $||\Theta_n f - f||_v \leq \varepsilon$ for any fixed $n \in \mathbb{N}, n \geq n_0$.*

Definition 8. Define

$$A_0(G) := \{f \in C(\overline{G}); f_{|G} \in H(G),\ \forall \eta > 0 \exists N \in \mathbb{R}_+ : |f(z)| < \eta\ \forall z \in G, |z| \geq N\},$$

endowed with the sup-norm.

Now we extend $\Theta_n f$ continuously to $\overline{G}$ by taking $(\Theta_n f)(x) = f(x+\frac{1}{n})\sqrt[n]{\frac{1}{x+i}}$ for $x \in \mathbb{R}$.

Lemma 9. *For each $f \in Hv_0(G)$ and each $n \in \mathbb{N}$ we have $\Theta_n f \in A_0(G)$, i.e. there exists a linear mapping*

$$R_n : Hv_0(G) \to A_0(G), \quad R_n f = f_n\ \forall\ n \in \mathbb{N}.$$

Proof: Let $f \in Hv_0(G)$ and $n \in \mathbb{N}$ be fixed. Set $\varepsilon = \frac{1}{n}$. With condition (iv) for the weight v there exists $b = b(\frac{1}{n}) > 0$ with $v(z) \geq b$ for all $z \in G$ with $\mathrm{Im}z \geq \varepsilon$. Then for each $z \in G$, also $v(z+\frac{i}{n}) \geq b$ holds. Now fix $\eta > 0$. $f \in Hv_0(G)$ means that for $\tilde{\eta} := \eta \cdot b$ there exists $N > 0$ such that

$$|f(z+\frac{i}{n})|v(z+\frac{i}{n}) \leq \tilde{\eta}$$

for all $z \in G$ with $|z| \geq N$. Then for f_n and such a $z \in G$ the following estimate holds:

$$\begin{aligned} |f_n(z)| &= |f(z+\tfrac{i}{n})||\sqrt[n]{\tfrac{1}{z+i}}| \\ &= |f(z+\tfrac{i}{n})|v(z+\tfrac{i}{n})\tfrac{1}{v(z+\frac{i}{n})}|\sqrt[n]{\tfrac{1}{z+i}}| \\ &\leq \tilde{\eta}\cdot\tfrac{1}{b} = \eta, \end{aligned}$$

hence $f_n \in A_0(G)$.

Lemma 10. *The restriction mapping*

$$R : A_0(G) \to Hv_0(G), \;\; f \to f_{|G},$$

is well-defined and continuous.

Proof: Fix $f \in A_0(G)$. By condition (v), v is bounded, i.e. there exists $M > 0$ with $v(z) \leq M$ for all $z \in G$. Let $\eta > 0$ be arbitrary, but fixed. Set $\eta' := \frac{\eta}{M}$. For η' there exists $N > 0$ such that $|f(z)| < \eta'$ for all $z \in G$ with $|z| \geq N$. Then $v(z)|f(z)| < M\frac{\eta}{M} = \eta$ for all $z \in G$ with $|z| \geq N$. Define $L := N + 1$. By (ii) we can extend v continuously to $\overline{G}$ by putting $\tilde{v}(z) := v(z)$ for $z \in G$ and $\tilde{v}(z) := 0$ elsewhere. $\tilde{v}$ is uniformly continuous on $K := [-L, L] \times i[\delta, L]$ for each $\delta > 0$. f is bounded on K which means that there exists $S > 0$ such that $|f(z)| \leq S$ for all $z \in K$. For $\varepsilon := \frac{\eta}{S} > 0$ there exists $\delta > 0 : z, z' \in K, |z - z'| < \delta \Rightarrow |\tilde{v}(z) - \tilde{v}(z')| < \varepsilon$. We would like to show that $v(z)|f(z)| < \eta$ for all $z \notin K$. The desired inequality holds if $|z| \geq N + 1$. Let $z = x + iy \notin K$ and consider $0 < y < \delta$ and $|z| \leq N + 1$. We get $|x - z| = |x - x - iy| = |y| < \delta$ and $\tilde{v}(z) = \tilde{v}(z) - \tilde{v}(x) < \varepsilon = \frac{\eta}{S}$, hence $v(z)|f(z)| < \frac{\eta}{S}S = \eta$ for all $z \notin K$.

Lemma 11. *The sequence $(R_n)_n$ of linear mappings $R_n : Hv_0(G) \to A_0(G)$ is uniformly bounded.*

Proof: For $n \geq n_0$ large enough so that condition (iii) can be applied, we get

$$\begin{aligned} ||R_n f||_v &= ||f_n||_v = \sup_{z\in G} |f_n(z)|v(z) = \sup_{z\in G} |f(z + \frac{i}{n}) \sqrt[n]{\frac{1}{z+i}}|v(z) \\ &\leq \sup_{z\in G} |f(z + \frac{i}{n})|v(z + \frac{i}{n})| \sqrt[n]{\frac{1}{z+i}}| \\ &\leq ||f||_v. \end{aligned}$$

Definition 12. Let D be the open unit disc, $D := \{z \in \mathbb{C}; |z| < 1\}$. Define the *disc algebra*

$$A(D) := \{f \in C(\overline{D});\; f_{|D} \text{ is holomorphic}\},$$

and the space

$$A_0(D) := \{f \in A(D);\; f(1) = 0\}.$$

Because the polynomials are dense in the disc algebra one can write $A_0(D)$ as

$$A_0(D) = \overline{\text{span}}\{z^j - 1;\; j = 1, 2, ...\}.$$

Bockarev [3] showed in 1974:

Proposition 13. *(Bockarev) The disc algebra $A(D)$ has a Schauder basis and therefore the bounded approximation property.*

Proposition 14. *$A_0(D)$ has the bounded approximation property.*

Proof: By proposition 13, $A(D)$ has the bounded approximation property. $p : A(D) \to A_0(D), p(f) = f - f(1), f \in A(D)$, is a bounded projection onto $A_0(D)$. Because of this, $A_0(D)$ is complemented in the disc algebra and inherits the bounded approximation property from $A(D)$.

Proposition 15. *There exists an isometric isomorphism T between $A_0(G)$ and $A_0(D)$.*

Proof: Compare [10], p. 81. Define $\alpha : G \to D, \alpha(z) := \frac{z-i}{z+i}$ for $z \in G$. α is a linear fractional transformation of the upper half-plane G onto the unit disc D. The inverse mapping of α is $\beta : D \to G, \beta(w) := i\frac{1+w}{1-w}, w \in D$. For each $c \geq 0$, α maps the half plane $\text{Im}z > c$ onto the disc $\{w; \;\; |w - \frac{c}{1+c}| < \frac{1}{1+c}\}$, and α maps the line $\text{Im}z = c$ onto the circle $\{w; \;\; |w - \frac{c}{1+c}| = \frac{1}{1+c}\}$ with the point 1 deleted, also $\beta(1) = \infty$ and $\alpha(\infty) = 1$. Now we can define

$$T : A_0(G) \to A_0(D) \text{ as } Tf := f \circ \alpha, f \in A_0(G),$$

which is an isometric isomorphism from $A_0(G)$ onto $A_0(D)$.

From now on we are following a method of Lusky (see [8]) to construct a suitable commuting approximating sequence $(V_n)_{n\in\mathbb{N}}$, $V_n : Hv_0(G) \to Hv_0(G)$ such that $V_n - V_{n-1}$ factors uniformly through l_∞^m's.

Definition 16. Let $\mathcal{H}(D) := \{f : \overline{D} \to \mathbb{C}; f \text{ continuous}, f_{|D} \text{ harmonic}\}$ endowed with the sup-norm and let $f \in \mathcal{H}(D)$ have the Fourier series $f(re^{i\varphi}) = \sum_{k=-\infty}^{\infty} \alpha_k r^{|k|} e^{ik\varphi}$. Define $\tilde{V}_n : \mathcal{H}(D) \to \mathcal{H}(D)$ as

$$(\tilde{V}_n f)(re^{i\varphi}) := \sum_{|k|\leq 2^n} \alpha_k r^{|k|} e^{ik\varphi} + \sum_{2^n<|k|\leq 2^{n+1}} \frac{2^{n+1} - |k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi},$$

and $V_n : A_0(D) \to A_0(D)$ as

$$V_n f := \tilde{V}_n f - (\tilde{V}_n f)(1) \cdot z^{2^n}, f \in A_0(D).$$

Lemma 17. *For the Fourier series $f = \sum \alpha_k r^{|k|} e^{ik\varphi}$ we define the Cesàro means $\sigma_n : \mathcal{H}(D) \to \mathcal{H}(D)$ by $\sigma_n(f) := \sum_{|k|\leq 2^n} \frac{2^n-|k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi}$, cf. [4]. Then*

$$2\sigma_{n+1}(f) - \sigma_n(f) = \tilde{V}_n(f)$$

holds for each $n \in \mathbb{N}$.

Proof: By calculating we obtain

$$\begin{aligned}
&2\sigma_{n+1}(f) - \sigma_n(f) \\
= \;& 2 \sum_{|k|\leq 2^{n+1}} \frac{2^{n+1} - |k|}{2^{n+1}} \alpha_k r^{|k|} e^{ik\varphi} - \sum_{|k|\leq 2^n} \frac{2^n - |k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi} \\
= \;& \sum_{2^n<|k|\leq 2^{n+1}} \frac{2^{n+1} - |k|}{2^n} \alpha_k r^{|k|} e^{ik\varphi} + \sum_{|k|\leq 2^n} \left(\frac{2^{n+1} - |k|}{2^n} - \frac{2^n - |k|}{2^n} \right) \alpha_k r^{|k|} e^{ik\varphi}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{2^n<|k|\leq 2^{n+1}} \frac{2^{n+1}-|k|}{2^n}\alpha_k r^{|k|}e^{ik\varphi} + \sum_{|k|\leq 2^n}\left(\frac{2^{n+1}-2^n}{2^n}\right)\alpha_k r^{|k|}e^{ik\varphi} \\
&= \sum_{2^n<|k|\leq 2^{n+1}} \frac{2^{n+1}-|k|}{2^n}\alpha_k r^{|k|}e^{ik\varphi} + \sum_{|k|\leq 2^n}\alpha_k r^{|k|}e^{ik\varphi} \\
&= \tilde{V}_n(f).
\end{aligned}
$$

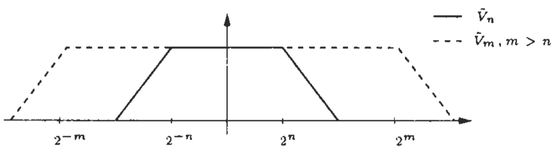
Lemma 18. *For $f \in A_0(D)$ and V_n defined as before, the following holds:*

(i) $\lim_{n\to\infty} V_n f = f$ *for each* $f \in A_0(D)$,

(ii) $\dim V_n A_0(D) < \infty$,

(iii) $V_n V_m = V_{\min(n,m)}$, *if* $n \neq m$.

Proof: (i) and (ii) follow immediately from the definition of V_n, respectively of $\tilde{V}_n$ and Lemma 17 because the Cesàro means are convergent to f in $A(\overline{D})$. To show (iii), we first prove $\tilde{V}_n\tilde{V}_m = \tilde{V}_{\min(n,m)}$, for $n \neq m$. For $m > n$, $\tilde{V}_n\tilde{V}_m = \tilde{V}_n$ follows directly from the definition. $\tilde{V}_n z^k = 0$ if $k \geq 2^{n+1}$, and $\tilde{V}_m z^k = \tilde{V}_n z^k = z^k$ if $k \leq 2^n < 2^m$.



For $n > m$ one can use the same arguments to get $\tilde{V}_n\tilde{V}_m = \tilde{V}_m$. To show the desired equation for V_nV_m, set $W_n(f) = -(\tilde{V}_n f)(1)z^{2^n}$. For $m > n$ we obtain:

$$
\begin{aligned}
V_nV_m(f) &= (\tilde{V}_n + W_n)(\tilde{V}_m + W_m)(f) \\
&= (\tilde{V}_n\tilde{V}_m + \tilde{V}_nW_m + W_n\tilde{V}_m + W_nW_m)(f) \\
&= \tilde{V}_n(f) - \tilde{V}_n((\tilde{V}_m f)(1)z^{2^m}) - \tilde{V}_n(\tilde{V}_m f)(1)z^{2^n} - W_n((\tilde{V}_m f)(1)z^{2^m}) \\
&= \tilde{V}_n(f) - (\tilde{V}_m f)(1)\tilde{V}_n(z^{2^m}) - (\tilde{V}_n f)(1)z^{2^n} + (\tilde{V}_m f)(1)(\tilde{V}_n(z^{2^m}))(1)z^{2^n} \\
&= \tilde{V}_n(f) + W_n(f) \\
&= V_n(f).
\end{aligned}
$$

In the case $m < n$ one uses the same arguments and obtains $V_nV_m = V_m$.

Lemma 19. *For trigonometric polynoms $\sum_k \alpha_k r^{|k|}e^{ik\varphi}$ define $P(\sum_k \alpha_k r^{|k|}e^{ik\varphi}) := \sum_{k\geq 0}\alpha_k r^{|k|}e^{ik\varphi}$, with generally unbounded P. Then $P(\tilde{V}_n - \tilde{V}_{n-1})(f) = e^{i2^n\varphi}\sigma_n(e^{-i2^n\varphi}f) - \frac{1}{2}e^{i2^{n-1}\varphi}\sigma_{n-1}(e^{-i2^{n-1}\varphi}f)$. Hence $P(\tilde{V}_n - \tilde{V}_{n-1})$ is a continuous linear operator and the same then holds for $P(V_n - V_{n-1})$.*

Proof: By some calculations we get

$$\begin{aligned}
&P(\tilde{V}_n - \tilde{V}_{n-1})(f) \\
= &\sum_{k=0}^{2^n} \alpha_k r^k e^{ik\varphi} - \sum_{k=0}^{2^{n-1}} \alpha_k r^k e^{ik\varphi} - \sum_{k=2^{n-1}+1}^{2^n} \frac{2^n - k}{2^{n-1}} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
= &\sum_{k=2^{n-1}+1}^{2^n} \alpha_k r^k e^{ik\varphi} - \sum_{k=2^{n-1}+1}^{2^n} \frac{2^n - k}{2^{n-1}} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
= &\sum_{k=2^{n-1}+1}^{2^n} \left(1 - \frac{2^n - k}{2^{n-1}}\right) \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
= &\sum_{k=2^{n-1}+1}^{2^n} \frac{k - 2^{n-1}}{2^{n-1}} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi}
\end{aligned}$$

and

$$\begin{aligned}
&e^{i2^n\varphi} \sigma_n(e^{-i2^n\varphi} f) - \frac{1}{2} e^{i2^{n-1}\varphi} \sigma_{n-1}(e^{-i2^{n-1}} \varphi) \\
= &\sum_{|k-2^n| \le 2^n} \frac{2^n - |k - 2^n|}{2^n} \alpha_k r^k e^{ik\varphi} - \frac{1}{2} \sum_{|k-2^{n-1}| \le 2^{n-1}} \frac{2^{n-1} - |k - 2^{n-1}|}{2^{n-1}} \alpha_k r^k e^{ik\varphi} \\
= &\sum_{0 \le k \le 2^{n+1}} \frac{2^n - |k - 2^n|}{2^n} \alpha_k r^k e^{ik\varphi} - \frac{1}{2} \sum_{0 \le k \le 2^n} \frac{2^{n-1} - |k - 2^{n-1}|}{2^{n-1}} \alpha_k r^k e^{ik\varphi} \\
= &\sum_{k=0}^{2^n} \frac{2^n - 2^n + k}{2^n} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^n - k + 2^n}{2^n} \alpha_k r^k e^{ik\varphi} \\
&- \sum_{k=0}^{2^{n-1}} \frac{2^{n-1} - 2^{n-1} + k}{2^n} \alpha_k r^k e^{ik\varphi} - \sum_{k=2^{n-1}+1}^{2^n} \frac{2^{n-1} - k + 2^{n-1}}{2^n} \alpha_k r^k e^{ik\varphi} \\
= &\sum_{k=0}^{2^n} \frac{k}{2^n} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
&- \sum_{k=0}^{2^{n-1}} \frac{k}{2^n} \alpha_k r^k e^{ik\varphi} - \sum_{k=2^{n-1}+1}^{2^n} \frac{2^n - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
= &\sum_{k=2^{n-1}+1}^{2^n} \left(\frac{k}{2^n} - \frac{2^n - k}{2^n}\right) \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
= &\sum_{k=2^{n-1}+1}^{2^n} \frac{k - 2^n + k}{2^n} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi} \\
= &\sum_{k=2^{n-1}+1}^{2^n} \frac{k - 2^{n-1}}{2^{n-1}} \alpha_k r^k e^{ik\varphi} + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k r^k e^{ik\varphi}.
\end{aligned}$$

Proposition 20. *$V_n - V_{n-1}$ factors uniformly through l_∞^m's on $A_0(D)$.*

Proof: By the definition of the Cesàro means, $||\sigma_n|| = 1$ holds for all $n \in \mathbb{N}$; again cf. [4]. With Lemma 17 we obtain $||\tilde{V}_n|| \leq 3$ for all $n \in \mathbb{N}$. Hence $(V_n)_n$ is uniformly bounded. $C(\partial D)$ is a $\mathcal{L}_\infty$-space, and it is well-known that $\mathcal{H}(D)$ is isometrically isomorphic to $C(\partial D)$. Hence $\mathcal{H}(D)$ is a $\mathcal{L}_\infty$-space. There exists $\lambda > 0$ such that for each $n \in \mathbb{N}$ there is $F \subset \mathcal{H}(D)$ with $\tilde{V}_{n+1}\mathcal{H}(D) \subset F$ and there is an isomorphism $\Phi : F \to l_\infty^M$ with $M = \dim F < \infty$ and $||\Phi|| \cdot ||\Phi^{-1}|| \leq \lambda$. Note that $A_0(D) \subset \mathcal{H}(D)$. Define $T_n : A_0(D) \to l_\infty^M$ by

$$T_n f := \Phi(V_{n+1} - V_{n-2})f,$$

and $S_n : l_\infty^M \to A_0(D)$ by

$$S_n g := P(V_n - V_{n-1})\Phi^{-1}g - (P(V_n - V_{n-1})\Phi^{-1}g)(1).$$

We have $\sup_n ||S_n|| < \infty$, $\sup_n ||T_n|| < \infty$ and

$$\begin{aligned} S_nT_n(f) &= S_n\Phi(V_{n+1} - V_{n-2})f \\ &= P(V_n - V_{n-1})(V_{n+1} - V_{n-2})f - (P(V_n - V_{n-1})(V_{n+1} - V_{n-2})f)(1) \\ &= P(V_n - V_{n-1})f - (P(V_n - V_{n-1})f)(1) \\ &= (V_n - V_{n-1})f \end{aligned}$$

where the last but one equality holds because of

$$\begin{aligned} (V_n - V_{n-1})(V_{n+1} - V_{n-2}) &= V_nV_{n+1} - V_nV_{n-2} - V_{n-1}V_{n+1} + V_{n-1}V_{n-2} \\ &= V_n - V_{n-2} - V_{n-1} + V_{n-2} \\ &= V_n - V_{n-1}. \end{aligned}$$

4. Proof of Theorem 1

Collecting the results of Section 3 we can now prove Theorem 1. First we give an overview of the operators defined before:

$$Hv_0(G) \xrightarrow{R_n} A_0(G) \xrightarrow{T} A_0(D) \xrightarrow{V_n} A_0(D) \xrightarrow{T^{-1}} A_0(G) \xrightarrow{R} Hv_0(G).$$

For a suitable sequence $(m_n)_{n\in\mathbb{N}}$ of indices we can assume without loss of generality:

$$(*)\ R_{m_n}RT^{-1}(r^{|k|}e^{ik\varphi} - 1) = T^{-1}(r^{|k|}e^{ik\varphi} - 1)\ \forall\ |k| \leq 2^{n+1}.$$

If $(*)$ is not true, replace R_{m_n} by

$$\begin{aligned} \tilde{R}_{m_n} &:= R_{m_n}(\mathrm{id} - P_n) + R^{-1}P_n \\ &= (R^{-1} - R_{m_n})P_n + R_{m_n}, \end{aligned}$$

with $E_n := \mathrm{span}\{RT^{-1}(r^{|k|}e^{ik\varphi} - 1);\ \ |k| \leq 2^{n+1}\}$, $E_n \subset Hv_0(G)$ and $P_n : Hv_0(G) \to E_n$ a bounded projection. Then

$$\tilde{R}_{m_n}RT^{-1} = (R^{-1} - R_{m_n})RT^{-1} + R_{m_n}RT^{-1} = T^{-1}$$

holds on E_n, but we have to show that $\tilde{R}_{m_n}$ is uniformly bounded. By Corollary 7, one can choose $m_1 < m_2 < \dots$ with

$$||R_{m_n}RT^{-1}(r^{|k|}e^{ik\varphi}-1) - T^{-1}(r^{|k|}e^{ik\varphi}-1)|| \leq \frac{1}{n2^{n+2}||P_n||w}$$

for all $|k| \leq 2^{n+1}$, where $w := ||R^{-1}{}_{|E_n}||$. By the definition of $\tilde{R}_{m_n}$ we obtain

$$||\tilde{R}_{m_n} - R_{m_n}|| = ||(R^{-1} - R_{m_n})P_n||.$$

Let $x \in E_n$ with $||x||_v = 1$. One can write x as

$$x := \sum_{|k|\leq 2^{n+1}} \alpha_k RT^{-1}(r^{|k|}e^{ik\varphi}-1).$$

With $U := (R^{-1} - R_{m_n})P_n$ one gets

$$||Ux||_v \leq \sum_{|k|\leq 2^{n+1}} |\alpha_k| \cdot ||URT^{-1}(r^{|k|}e^{ik\varphi}-1)||_v.$$

Define $F_n := \operatorname{span}\{(r^{|k|}e^{ik\varphi}-1);\ \ |k| \leq 2^{n+1}\}$. Then $F_n \subset A_0(D)$, $RT^{-1}F_n = E_n$ and $||(RT^{-1}{}_{|F_n})^{-1}|| \leq w\ ||T||$ holds. Set $W := (RT^{-1}{}_{|F_n})^{-1}$ and note and

$$Wx = \sum_{|k|\leq 2^{n+1}} |\alpha_k|(r^{|k|}e^{ik\varphi}-1).$$

Here the Fourier coefficients can be estimated as follows:

$$|\alpha_k| \leq ||Wx|| \leq ||W|| \cdot ||x||_v = ||W|| \leq w\ ||T||.$$

Putting the estimates together we obtain

$$\begin{aligned}
||\tilde{R}_{m_n} - R_{m_n}|| &= \sup\{||Ux||_v; ||x||_v = 1\} \\
&\leq \sum_{|k|\leq 2^{n+1}} |\alpha_k| \cdot ||URT^{-1}(r^{|k|}e^{ik\varphi}-1)||_v \\
&\leq 2^{n+2}w||T|| \cdot ||T^{-1}(r^{|k|}e^{ik\varphi}-1) - R_{m_n}RT^{-1}(r^{|k|}e^{ik\varphi}-1)|| \\
&\leq \frac{||T||}{n||P_n||}.
\end{aligned}$$

Now define $\hat{V}_n : Hv_0(G) \to Hv_0(G)$ by

$$\hat{V}_n := RT^{-1}V_nTR_{m_n}.$$

We claim that $\hat{V}_n$ is a commuting approximating sequence with $\hat{V}_n\hat{V}_m = \hat{V}_{\min(n,m)}$ for $n \neq m$, $\dim \hat{V}_nHv_0(G) < \infty$ and $\lim_{n\to\infty} \hat{V}_nf = f$ for $f \in Hv_0(G)$. Let $n > m$; then we have:

$$\begin{aligned}
\hat{V}_n\hat{V}_m &= RT^{-1}V_nTR_{m_n}RT^{-1}V_mTR_{m_m} \\
&= RT^{-1}V_nTT^{-1}V_mTR_{m_m} \\
&= RT^{-1}V_nV_mTR_{m_m} \\
&= RT^{-1}V_mTR_{m_m} \\
&= \hat{V}_m.
\end{aligned}$$

This holds because of $(*)$ and because TT^{-1} is the identity on $A_0(D)$. If $n < m$ we obtain $\hat{V}_n\hat{V}_m = \hat{V}_n$ by the same arguments. In Proposition 20 we showed that there exist k_n, $T_n : A_0(D) \to l_\infty^{k_n}$ and $S_n : l_\infty^{k_n} \to A_0(D)$ with $\sup_n ||S_n|| < \infty$, $\sup_n ||T_n|| < \infty$ and $S_nT_n = V_n - V_{n-1}$. Set

$$\begin{aligned} \hat{T}_n : Hv_0(G) \to l_\infty^{k_n}, \quad \hat{T}_n &:= \quad T_nTR_{m_n}, \\ \hat{S}_n : l_\infty^{k_n} \to Hv_0(G), \quad \hat{S}_n &:= \quad RT^{-1}S_n. \end{aligned}$$

With $(*)$ and the definition of V_n it follows that

$$(**)\ V_nTR_{m_j} = V_nTR_{m_n}$$

holds for all $j \geq n$ since $V_nTR_{m_n}RT^{-1}(r^{|k|}e^{ik\varphi}-1) = V_nTT^{-1}(r^{|k|}e^{ik\varphi}-1) = V_n(r^{|k|}e^{ik\varphi}-1)$ for each $|k| \leq 2^{n+1}$. Note that $\sup_n ||\hat{S}_n|| < \infty$, $\sup_n ||\hat{T}_n|| < \infty$ and by $(**)$

$$\begin{aligned} \hat{S}_n\hat{T}_n &= \hat{S}_n(T_nTR_{m_n}) \\ &= RT^{-1}S_nT_nTR_{m_n} \\ &= RT^{-1}(V_n - V_{n-1})TR_{m_n} \\ &= (RT^{-1}V_n - RT^{-1}V_{n-1})TR_{m_n} \\ &= RT^{-1}V_nTR_{m_n} - RT^{-1}V_{n-1}TR_{m_{n-1}} \\ &= \hat{V}_n - \hat{V}_{n-1}. \end{aligned}$$

We have constructed a commuting approximating sequence $\hat{V}_n$ such that $\hat{V}_n - \hat{V}_{n-1}$ factors uniformly through l_∞^m's. With Theorem 3 it follows that $Hv_0(G)$ has a basis.

5. Examples

Example 21. Let G be the upper half-plane and $v : G \to \mathbb{R}$ be defined by $v(z) := (\mathrm{Im}z)^r$ for $\mathrm{Im}z \leq 1$ and $v(z) := 1$ elsewhere, $r > 0$. v satisfies the conditions (i) - (v). Hence $Hv_0(G)$ has a basis.

Example 22. Let G be the upper half-plane and $v : G \to \mathbb{R}$ be defined by $v(z) := \exp(-1/(\mathrm{Im}z)^2)$. It is easy to see that v satisfies conditions (i)-(v). Hence $Hv_0(G)$ has a basis.

Example 23. Let G be the upper half-plane and $v : G \to \mathbb{R}$ be defined by $v(z) := \mathrm{Im}z$. v satisfies conditions (i)-(iv), but v is not bounded. But $Hv_0(G)$ has the bounded approximation property.

Proof: The idea of this construction goes back to Stanev [11]. Let the weight w on the unit disc D be defined by $w(\delta) := (1 - |\delta|^2)$. w is radial and $\lim_{|\delta|\to 1} w(\delta) = 0$. Hence $Hw_0(D)$ has the bounded approximation property [2]. For $f \in Hw_0(D)$ we define the operator $\tilde{T} : Hw_0(D) \to Hv_0(G)$, $\tilde{T}f(z) = (f \circ \tilde{\beta})(z) \cdot (\frac{4}{(1-iz)^2})$, $z \in G$ with $\tilde{\beta}(z) = \frac{1+iz}{1-iz}$ for $z \in G$. $\tilde{\beta}$ maps the upper half-plane G onto the unit disc D. The operator $\tilde{T}$ is a topological isomorphism from $Hw_0(D)$ onto $Hv_0(G)$ [11].

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