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AN ALTERNATIVE PROOF OF A PROPERTY OF THE RADON TRANSFORM ON THE HARDY SPACE H^1

C. CARTON-LEBRUN

1. INTRODUCTION

Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and R_ω be the Radon transform relatively to $\omega \in S^{n-1}$, i.e. the integral over almost every hyperplane $< x,\omega > = t$.

In this note, we give a non-constructive proof of the following known property :

R_ω is a bounded surjective operator from the Hardy space $H^1(\mathbb{R}^n)$ onto $H^1(\mathbb{R})$.

The boundedness assertion is contained in [1] where it results from an atomic decomposition of H^1 functions. It is also included in [3] where it occurs as a corollary of an identity relating the Radon and Riesz transforms. A constructive proof of the surjectivity of R_ω is contained in [3] too.

The alternative proof we give here is essentially based on duality arguments. In particular, it brings out the fact that the dual operator of $R_\omega : H^1(\mathbb{R}^n) \to H^1(\mathbb{R})$ is $B_\omega : g \to g(< .,\omega >) : BMO(\mathbb{R}) \to BMO(\mathbb{R}^n)$.

2. PROOF

By Fubini theorem, we see that

$$(\star) \quad \int_{\mathbb{R}} g.R_\omega f \, dt = \int_{\mathbb{R}^n} B_\omega g.f \, dx$$

holds for every $f \in \mathcal{S}_o(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R})$, where

$$\mathcal{S}_o(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : \int f \, dx = 0\} \quad \text{and } B_\omega g = g(< .,\omega >).$$

On the other hand, we notice the following facts :

a. $R_\omega f \in {}_o(\mathbb{R})$ whenever $f \in \mathcal{S}_o(\mathbb{R}^n)$. This is an easy consequence of the identity relating the Fourier transforms of f and $R_\omega f$.

b. $\mathcal{S}_o(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$, $n \geqslant 1$. This results from lemma 1.5 of [2]. Moreover, $\mathcal{S}_o(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ since it contains the dense subspace $H^1_o(\mathbb{R}^n)$ considered in [4], p 231.

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c. B_ω is a one-to-one bicontinuous map from $BMO(\mathbb{R})$ into $BMO(\mathbb{R}^n)$.
Indeed, if $m_Q g = \int_Q g\,dx$, a straightforward comparison of the BMO norms of g and $g(<.,\omega>)$, defined as $\sup(m_Q|g - m_Q g|)$ over all cubes $Q \subset \mathbb{R}^n$, with n = 1 and n > 1 respectively, yields the required assertion.

From the above remarks, we first deduce that

$$\|R_\omega f\|_{H^1(\mathbb{R})} \leq C\,\|f\|_{H^1(\mathbb{R}^n)} \qquad \forall f \in \mathcal{S}_o(\mathbb{R}^n),$$

which implies that R_ω has a bounded extension on the whole of $H^1(\mathbb{R}^n)$. Moreover, owing to the boundedness of R_ω as an operator from $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R})$, this extended operator coïncides with the usual Radon transform defined as an integral.

From (*), we next conclude that B_ω is the dual operator of R_ω. The surjectivity thesis is then a consequence of the Banach closed range theorem ([5], corollary 1, p 208).

Remark. If $N(R_\omega)$ denotes the null-space of $R_\omega : H^1(\mathbb{R}^n) \to H^1(\mathbb{R})$, we notice that

$$N(R_\omega) = \{f \in H^1(\mathbb{R}^n) : (g(<.,\omega>), f) = 0 \quad \forall\, g \in BMO(\mathbb{R})\},$$

$$N(R_\omega)^{\perp} = \{g(<.,\omega>) : g \in BMO(\mathbb{R})\}.$$

3. REFERENCES

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Université de l'Etat à MONS
Service de Mathématique
Avenue du Champ de Mars, 24
B - 7000 - MONS
BELGIQUE