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A Real Analytic Schwartz' Kernels Theorem

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Abstract

In this short paper, we study a few topological properties of the sheaf of real analytic functions on a real analytic manifold M. In particular, we show that its topological Poincaré-Verdier dual is the sheaf of hyperfunction densities on M. We also prove that if N is a second real analytic manifold, then the continuous cohomological correspondences between the sheaf of real analytic functions on M and the sheaf of hyperfunctions on N are given by integral transforms whose kernels are hyperfunction forms on $M \times N$ of a suitable kind. This result may be viewed as a real analytic analogue of the well-known kernels theorem of Schwartz.

Introduction

Let M be a real analytic manifold of dimension m and let X be a complexification of M. Denote or_M the orientation sheaf of M and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. A classical pure codimensionality theorem due to M. Sato sates that all the cohomology sheaves of the complex

$$\mathrm{or}_M \otimes \mathrm{R}\Gamma_M(\mathcal{O}_X)$$

vanish except for the m-th one. This non-vanishing cohomology sheaf is then defined to be the sheaf $\mathcal{B}_M$ of hyperfunctions on M.

This approach is at first glance completely different from the one followed by L. Schwartz to construct the sheaf of distributions on a smooth manifold. Recall that, if M is a smooth manifold of dimension m, one defines the sheaf $\mathcal{D}b$ of distributions on M by duality through the formula

$$\mathcal{D}b(U) = \mathrm{L}\,(\Gamma_c(U; \mathrm{or}_M \otimes \mathcal{C}^m_\infty), \mathbb{C})$$

where $\mathcal{C}^m_\infty)$ is the sheaf of smooth m-forms on M and U is any open subset of M.

Of course, there are well-known ways to construct the sheaf of hyperfunctions on M by means of duality formulas. However, these constructions are not very functorial and rely on ad-hoc tricks.

In this paper, we will show that L. Schwartz approach works almost as is in the real analytic framework provided that we work at the cohomological level in a suitable category of topological sheaves.

More precisely, we will show that the topological Poincaré-Verdier dual of the topological sheaf $\mathcal{V}_M^\tau$ of real analytic densities on a real analytic manifold M is isomorphic the topological sheaf $\mathcal{B}_M^\tau$ of hyperfunctions; a result which entails that

$$\mathrm{R}\Gamma(U; \mathcal{B}_M^\tau) \simeq \mathrm{RL}\,(\mathrm{R}\Gamma_c(U; \mathcal{V}_M^\tau), \mathbb{C}^\tau).$$

This result will appear as an easy consequence of the results in [7]. Moreover, the method followed in the proof will also allow us to recover easily Sato's pure codimensionality theorem.

The second result established in this paper is also a consequence of the results in [7] and generalizes to a real analytic situation the kernels theorem of L. Schwartz. A well-known result of sheaf theory adapted to the case of topological sheaves states that for any topological sheaf F on M and any topological sheaf G on N, we have

$$\mathrm{R}\Gamma(M \times N; \mathrm{R}\mathcal{L}\,(p_M^{-1}F, p_N^!G)) \simeq \mathrm{RL}\,(\mathrm{R}\Gamma_c(M; F), \mathrm{R}\Gamma(N; G)).$$

Here, we will replace F by $\mathcal{A}_M^\tau$ and G by $\mathcal{B}_N^\tau$ and show that the corresponding kernel sheaf

$$\mathrm{R}\mathcal{L}\,(p_M^{-1}F, p_N^!G).$$

is canonically isomorphic to

$$p_M^{-1}\mathcal{V}_M^\tau \,\hat{\otimes}_{p_M^{-1}\mathcal{A}_M^\tau}\, \mathcal{B}_{M\times N}^\tau.$$

In other words, we will show that continous cohomological correspondences between the sheaf of real analytic functions on M and the sheaf of hyperfunctions on N are given by integral transforms whose kernels are hyperfunction forms on $M \times N$ of a suitable kind.

The paper is divided in four sections.

The first two are of introductory nature and are devoted to short surveys on topological sheaves and on topological properties of the sheaves of holomorphic functions. In the third section we study the topological properties of the sheaves of real analytic functions and hyperfunctions and we establish the duality theorem. The last section deals with the kernels theorem.

1 Topological sheaves

The naive point of view one can adopt if one wants to deal with topological sheaves is to consider the category $\mathcal{T}c$ of locally convex topological vector spaces and to work with the category

$$Shv(X; \mathcal{T}c)$$

of sheaves on X with values in $\mathcal{T}c$.

If one follows this approach, one encounters two main difficulties.

The first one is that the category $\mathcal{T}c$ is not abelian. Hence, we cannot use the usual formalism to do homological algebra in $\mathcal{T}c$. The category $\mathcal{T}c$ is however not too far from an abelian one and we can deal with its homological algebra through the theory of quasi-abelian categories developed in [8] (see [6]).

The second difficulty is more serious and is related to the fact that inductive limits are not exact in $\mathcal{T}c$. To work around it, we proposed in [7] to replace the category $\mathcal{T}c$ with the category $\mathcal{I}nd(\mathcal{B}an)$ formed by the ind-objects of the category of Banach spaces. As a matter of fact, it follows from [8] that most of the well-known results of the cohomological theory of abelian sheaves hold for sheaves with values in this category (Poincaré-Verdier duality included). Moreover, this category is sufficiently related to $\mathcal{T}c$ so that we can use it to keep track of the topological information carried by the sheaves we encounter in algebraic analysis.

As a help to the reader, and to fix the notations, let us briefly recall the main facts concerning quasi-abelian sheaves and more specifically sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$.

The central notion is that of a quasi-abelian category (i.e. an additive category with kernels and cokernels such that the push-forward (resp. the pull-back) of a kernel (resp. a cokernel) is also a kernel (resp. a cokernel)). Let $\mathcal{E}$ be such a category. A morphism of $\mathcal{E}$ is said to be strict if its coimage is canonically isomorphic to its image. A complex

$$\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \cdots$$

of $\mathcal{E}$ is said to be strictly exact in degree k if d^{k-1} is strict and $\operatorname{Ker} d^k = \operatorname{Im} d^{k-1}$. Localizing the triangulated category $K(\mathcal{E})$ of complexes "modulo homotopy" by the null system formed by the complexes which are strictly exact in every degree gives us the derived category $D(\mathcal{E})$. This category has two canonical t-structures. Here, we will only use the left one. Its heart $\mathcal{LH}(\mathcal{E})$ is formed by the complexes of the form

$$0 \to X^{-1} \xrightarrow{d^{-1}} X^0 \to 0$$

where d^{-1} is a monomorphism. The cohomology functor

$$LH^k : D(\mathcal{E}) \to \mathcal{LH}(\mathcal{E})$$

sends the complex X to the complex

$$0 \to \operatorname{Coim} d^{k-1} \to \operatorname{Ker} d^k \to 0$$

with $\operatorname{Ker} d^k$ in degree 0. In [8], it was shown that in most problems of homological algebra we may replace the quasi-abelian category $\mathcal{E}$ by the abelian category $\mathcal{LH}(\mathcal{E})$ without loosing any information. It was also shown there that if $\mathcal{E}$ is elementary (i.e. if it has a small strictly generating set formed by tiny projective objects), then the sheaves with values in $\mathcal{E}$ share most of the usual properties of sheaves of abelian groups (including Poincaré-Verdier duality). If $\mathcal{E}$ has moreover a closed structure given by an internal tensor product and an internal Hom functor satisfying some natural assumptions, then Künneth theorem holds for sheaves with values in $\mathcal{E}$.

One checks easily (see e.g. [4]) that the category $\mathcal{B}an$ is quasi-abelian and that for any set I, the space $l^1(I)$ (resp. $l^\infty(I)$) of summable (resp. bounded) sequences of $\mathbb{C}$ indexed by I is projective (resp. injective) in $\mathcal{B}an$. Using these spaces, one can prove that $\mathcal{B}an$ has enough injective and projective objects. As is well-kown, the category $\mathcal{B}an$ has also a canonical structure of closed additive category. This structure is given by a right exact tensor product

$$\hat{\otimes} : \mathcal{B}an \times \mathcal{B}an \to \mathcal{B}an$$

and a left exact internal Hom

$$\mathrm{L} : \mathcal{B}an^{\mathrm{op}} \times \mathcal{B}an \to \mathcal{B}an.$$

Denoting $\hat{\otimes}^L$ the left derived functor of $\hat{\otimes}$ and RL the right derived functor of L, we have the adjunction formula

$$\mathrm{RHom}\,(E \,\hat{\otimes}^L F, G) \simeq \mathrm{RHom}\,(E, \mathrm{RL}\,(F, G)).$$

Let $\mathfrak{U}$, $\mathfrak{V}$ be two universes such that $\mathfrak{V} \ni \mathfrak{U}$. Denote $\mathcal{B}an_{\mathfrak{U}}$ the category formed by the Banach spaces which belong to $\mathfrak{U}$ and consider the category

$$\mathcal{I}nd_{\mathfrak{V}}(\mathcal{B}an_{\mathfrak{U}})$$

of ind-objects of $\mathcal{B}an_{\mathfrak{U}}$. Recall that the objects of $\mathcal{I}nd_{\mathfrak{V}}(\mathcal{B}an_{\mathfrak{U}})$ are functors

$$E : \mathcal{I} \to \mathcal{B}an_{\mathfrak{U}}$$

where $\mathcal{I}$ is a $\mathfrak{V}$-small filtering category and that if

$$E : \mathcal{I} \to \mathcal{B}an_{\mathfrak{U}}, \qquad F : \mathcal{J} \to \mathcal{B}an_{\mathfrak{U}}$$

are two such functors, then

$$\mathrm{Hom}_{\,\mathcal{I}nd_{\mathfrak{V}}(\mathcal{B}an_{\mathfrak{U}})}(E, F) = \varprojlim_{i \in \mathcal{I}} \varinjlim_{j \in \mathcal{J}} \mathrm{Hom}_{\,\mathcal{B}an_{\mathfrak{U}}}(E(i), F(j)).$$

For further details on ind-objects, we refer the reader to classical sources (such as [1, 2]) and to [5]. Following the standard usage and to avoid confusions, we will denote

$$\text{“}\varinjlim_{i \in \mathcal{I}}\text{”}\, E(i)$$

the functor $E : \mathcal{I} \to \mathcal{B}an_{\mathfrak{U}}$ considered as an object of $\mathcal{I}nd_{\mathfrak{V}}(\mathcal{B}an_{\mathfrak{U}})$. Similarly, we denote “$X$” the ind-object associated to the $\mathfrak{U}$-Banach space X. In other words, we set

$$\text{“}X\text{”} = \text{“}\varinjlim_{i \in \mathcal{I}}\text{”}\, C(i)$$

where $\mathcal{I}$ is a one point category and $C : \mathcal{I} \to \mathcal{B}an_{\mathfrak{U}}$ is the constant functor with value X.

Note that from a homological point of view the category $\mathcal{I}nd_{\mathfrak{U}}(\mathcal{B}an_{\mathfrak{U}})$ is very close to the category of qB-spaces introduced and studied by Waelbroeck (see [9]). This category is however, a little bit different from the category $\mathcal{I}nd_{\mathfrak{V}}(\mathcal{B}an_{\mathfrak{U}})$ considered here.

In the rest of this paper, we will not make the universes $\mathfrak{U}$, $\mathfrak{V}$ explicit in our notations since this is not really necessary for a clear understanding. So we will simply denote $\mathcal{I}nd(\mathcal{B}an)$ the category

$$\mathcal{I}nd_{\mathfrak{V}}(\mathcal{B}an_{\mathfrak{U}}).$$

Thanks to a result in [8], we know that this category is an elementary closed quasi-abelian category. It follows that sheaves with values in it share most of the usual properties of abelian sheaves (including Künneth Theorem and Poincaré-Verdier duality). We will call them *topological sheaves*.

In $\mathcal{I}nd(\mathcal{B}an)$, the internal tensor product

$$\hat{\otimes} : \mathcal{I}nd(\mathcal{B}an) \times \mathcal{I}nd(\mathcal{B}an) \to \mathcal{I}nd(\mathcal{B}an)$$

and the internal Hom functor

$$\mathrm{L} : (\mathcal{I}nd(\mathcal{B}an))^{\mathrm{op}} \times \mathcal{I}nd(\mathcal{B}an) \to \mathcal{I}nd(\mathcal{B}an)$$

are characterized by

$$(\text{“}\varinjlim_{i\in I}\text{”}\, E_i) \,\hat{\otimes}\, (\text{“}\varinjlim_{j\in J}\text{”}\, F_j) = \varinjlim_{i\in I} \varinjlim_{j\in J} \text{“}E_i \,\hat{\otimes}\, F_j\text{”}$$

and

$$\mathrm{L}\,(\text{“}\varinjlim_{i\in I}\text{”}\, E_i, \text{“}\varinjlim_{j\in J}\text{”}\, F_j) = \varprojlim_{i\in I} \varinjlim_{j\in J} \text{“}\mathrm{L}\,(E_i, F_j)\text{”}.$$

The internal tensor product (resp. internal Hom functor, external tensor product) for sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$ will be denoted by $\hat{\otimes}$ (resp. $\mathcal{L}$, $\hat{\boxtimes}$) and we will use the classical notations for the other usual functors of sheaf theory. In particular, $Rf_!$ (resp. $f^!$) will denote the direct (resp. inverse) image with proper support.

To make a distinction between a classical sheaf of $\mathbb{C}$-vector spaces and its topological analogue, we will append the superscript τ to the latter one. For example $\mathbb{C}^\tau_X$ will denote the constant topological sheaf sheaf with fiber “$\mathbb{C}$” and

$$\omega^\tau_X = a^!_X \mathbb{C}^\tau_{\{\mathrm{pt}\}} \qquad (a_X : X \to \{\mathrm{pt}\})$$

will denote the dualizing complex for topological sheaves. Following our conventions, the Poincaré-Verdier dualizing functor for topological sheaves is the functor

$$\mathrm{D}(\cdot) = \mathrm{R}\mathcal{L}\,(\cdot, \omega^\tau_X).$$

A link between the category $\mathcal{T}c$ and the category $\mathcal{I}nd(\mathcal{B}an)$ is given by the functor $\mathrm{IB} : \mathcal{T}c \to \mathcal{I}nd(\mathcal{B}an)$ defined by setting

$$\mathrm{IB}(E) = \text{“}\varinjlim_{B\in\mathcal{B}_E}\text{”}\, \widehat{E}_B$$

where $\mathcal{B}_E$ is the set of absolutely convex bounded subsets of E and E_B the linear hull of B. This functor was studied in [7]. Although it is not fully faithful, we have

$$\mathrm{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(\mathrm{IB}(E), \mathrm{IB}(F)) \simeq \mathrm{Hom}_{\mathcal{T}c}(E, F)$$

and

$$\mathrm{IB}(\mathrm{L}_b(E, F)) \simeq \mathrm{L}\,(\mathrm{IB}(E), \mathrm{IB}(F)).$$

if E is bornological and F complete. Moreover, IB is compatible with projective limits of filtering projective systems of complete spaces and with complete inductive limits of injective inductive systems of Fréchet spaces indexed by $\mathbb{N}$. It follows from these properties that if X is a topological space with a countable basis and if F is a presheaf of Fréchet spaces on X which is a sheaf of vector spaces, then

$$U \mapsto \mathrm{IB}(F(U)) \qquad (U \text{ open of } X)$$

is a sheaf with values in $\mathcal{I}nd(\mathcal{B}an)$. This result allows us to associate topological sheaves to many of the sheaves we encounter in algebraic analysis. This is in particular the case for the sheaf Ω_X^p of holomorphic p-forms and for the sheaf $\mathcal{C}_{\infty,X}^{(p,q)}$ of smooth (p,q)-forms on a complex analytic manifold X.

To end this section let us recall the two acyclicity results established in [7].

The first one is related to a result of Palamodov for $\mathcal{T}c$ (see [3]) and states that if E is a DFN space and if F is a Fréchet space, then both $LH^k(\mathrm{RHom}\,(\mathrm{IB}(E), \mathrm{IB}(F)))$ and $LH^k(\mathrm{RL}\,(\mathrm{IB}(E), \mathrm{IB}(F)))$ are 0 for $k \neq 0$.

The second one states that if E and F are objects of $\mathcal{I}nd(\mathcal{B}an)$ with E nuclear, then $E \mathbin{\hat{\otimes}}^L F \simeq E \mathbin{\hat{\otimes}} F$.

2 Topological properties of $\mathcal{O}_X$

Let X be a complex analytic manifold and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. As was explained in the preceding section, we can associate to $\mathcal{O}_X$ the topological sheaf

$$\mathcal{O}_X^{\tau} = \mathrm{IB}(\mathcal{O}_X).$$

This sheaf was studied in [7]. Let us recall some of its properties.

2.1 Cartan's Theorem B

Let X be a complex analytic manifold and let U be an open subset of X such that $H^k(U, \mathcal{O}_X) \simeq 0$ $(k \neq 0)$. Then, we have the canonical isomorphism

$$\mathrm{R}\Gamma(U, \mathcal{O}_X^{\tau}) \simeq \Gamma(U, \mathcal{O}_X^{\tau}) \simeq \mathrm{IB}(\mathcal{O}_X(U)).$$

It follows that, if X is a Stein manifold and K is a holomorphically convex compact subset of X, then

$$\mathrm{R}\Gamma(K, \mathcal{O}_X^{\tau}) \simeq \Gamma(K, \mathcal{O}_X^{\tau}) \simeq \mathrm{IB}(\mathcal{O}_X(K)).$$

2.2 Holomorphic Künneth Theorem

Let X and Y be complex analytic manifolds. Then, we have the canonical isomorphism

$$\mathcal{O}_X^\tau \hat{\boxtimes}^L \mathcal{O}_Y^\tau \simeq \mathcal{O}_{X\times Y}^\tau.$$

Consequently, if A, B are subsets of X and Y, then

$$\mathrm{R}\Gamma_c(A\times B, \mathcal{O}_{X\times Y}^\tau) \simeq \mathrm{R}\Gamma_c(A, \mathcal{O}_X^\tau) \hat{\otimes}^L \mathrm{R}\Gamma_c(B; \mathcal{O}_Y^\tau).$$

2.3 Holomorphic Poincaré duality

Let X be a complex analytic manifold of dimension d_X and let Ω_X^τ be the topological sheaf associated to the sheaf Ω_X of holomorphic forms of degree d_X. Then, we have the canonical isomorphism

$$\mathrm{D}(\mathcal{O}_X^\tau) \simeq \Omega_X^\tau[d_X].$$

2.4 Holomorphic Schwartz' kernels theorem

Let X, Y be complex analytic manifolds of dimension d_X, d_Y and denote

$$\Omega_{X\times Y}^{(d_X,0)\tau}$$

the topological sheaf associated to the sheaf of holomorphic forms of degree d_X in the X variables and 0 in the Y variables. Then, we have the canonical isomorphism

$$\Omega_{X\times Y}^{(d_X,0)\tau}[d_X] \simeq \mathrm{R}\mathcal{L}\,(q_X^{-1}\mathcal{O}_X^\tau, q_Y^!\mathcal{O}_Y^\tau).$$

This isomorphism entails that

$$\mathrm{R}\Gamma(X\times Y; \Omega_{X\times Y}^{(d_X,0)\tau}[d_X]) \simeq \mathrm{RL}\,(\mathrm{R}\Gamma_c(X; \mathcal{O}_X^\tau), \mathrm{R}\Gamma(Y; \mathcal{O}_Y^\tau))).$$

3 Topological properties of $\mathcal{A}_M$

Let M be a real analytic manifold of dimension m and let X be a Stein complexification of M.

Definition 3.0.1. We denote $\mathcal{A}_M^\tau$ the restriction to M of the topological sheaf $\mathcal{O}_X^\tau$.

Remark 3.0.2. The sheaf of $\mathbb{C}$-vector spaces associated to $\mathcal{A}_M^\tau$ is of course the sheaf $\mathcal{A}_M$ of real analytic functions on M, hence the notation. If K is a compact subset of M, then

$$\Gamma(K; \mathcal{A}_M^\tau) \simeq \Gamma(K; \mathcal{O}_X^\tau)$$

is isomorphic to $\mathrm{IB}(\mathcal{O}_X(K))$ where $\mathcal{O}_X(K)$ is endowed with its classical DFN topology. However, if U is an open subset, we do not know the relations between the ind-Banach space

$$\Gamma(U; \mathcal{A}_M^\tau)$$

and the locally convex topological vector spaces obtained by endowing $\mathcal{A}_M(U)$ with its various classical topologies.

Proposition 3.0.3.

(a) The topological Poincaré-Verdier dual

$$\mathrm{D}(\mathcal{A}_M^\tau) = \mathrm{R}\mathcal{L}(\mathcal{A}_M^\tau; \omega_M^\tau)$$

is concentrated in degree 0.

(b) We have the canonical isomorphism

$$\mathrm{D}(\mathcal{A}_M^\tau) \simeq \mathrm{R}\Gamma_M(\Omega_X^\tau)[m].$$

Proof. (a) Let K be a compact subset of M and denote

$$i_K : K \to M$$

the canonical inclusion. We have

$$i_K^! \mathrm{D}(\mathcal{A}_M^\tau) \simeq \mathrm{D}(i_K^{-1}\mathcal{A}_M^\tau).$$

Hence,

$$\mathrm{R}\Gamma_K(M; \mathrm{D}(\mathcal{A}_M^\tau)) \simeq \mathrm{RL}\,(\mathrm{R}\Gamma(K; \mathcal{A}_M^\tau), \mathbb{C}).$$

Since

$$\mathrm{R}\Gamma(K; \mathcal{A}_M^\tau) \simeq \mathrm{R}\Gamma(K; \mathcal{O}_X^\tau)$$

and K is holomorphically convex in X, it follows from the results recalled in Subsection 2.1 that this complex is isomorphic to

$$\mathrm{IB}(\mathcal{O}_X(K)).$$

Morover, the space $\mathcal{O}_X(K)$ being DFN, the first acyclicity theorem recalled in Section 1 shows that

$$\mathrm{RL}\,(\mathrm{IB}(\mathcal{O}_X(K)), \mathbb{C}) \simeq \mathrm{IB}(\mathrm{L}_b(\mathcal{O}_X(K), \mathbb{C})).$$

In particular,

$$\mathrm{R}\Gamma_K(M; \mathrm{D}(\mathcal{A}_M^\tau))$$

is concentrated in degree 0.

Now, let U be a relatively compact open subset of X. Using the distinguished triangle

$$\mathrm{R}\Gamma_U(M; \mathrm{D}(\mathcal{A}_M^\tau)) \to \mathrm{R}\Gamma_{\overline{U}}(M; \mathrm{D}(\mathcal{A}_M^\tau)) \to \mathrm{R}\Gamma(U; \mathrm{D}(\mathcal{A}_M^\tau)) \xrightarrow{+1}$$

we see that

$$\mathrm{R}\Gamma(U; \mathrm{D}(\mathcal{A}_M^\tau))$$

is also concentrated in degree 0. The conclusion follows.

(b) Denote

$$i_M : M \to X$$

the canonical inclusion. We know (see Subsection 2.3) that

$$\mathrm{D}(\mathcal{O}_X^\tau) \simeq \Omega_X^\tau[m].$$

Therefore,

$$i_M^! \Omega_X^\tau[m] \simeq i_M^! \mathrm{D}(\mathcal{O}_X^\tau) \simeq \mathrm{D}(i_M^{-1}\mathcal{O}_X^\tau) \simeq \mathrm{D}(\mathcal{A}_M^\tau).$$

□

Corollary 3.0.4. *The complex*

$$\mathrm{R}\Gamma_M(\mathcal{O}_X^\tau)$$

is concentrated in degree m.

Proof. Since Ω_X^τ is locally isomorphic to $\mathcal{O}_X^\tau$, this follows from the conjunction of parts (a) and (b) of the preceding proposition. □

Remark 3.0.5. Since the complex of sheaf of $\mathbb{C}$-vector spaces associated to $\mathrm{R}\Gamma_M(\mathcal{O}_X^\tau)$ is $\mathrm{R}\Gamma_M(\mathcal{O}_X)$, the preceding result contains Sato's pure codimensionality theorem which lies at the basis of the theory of hyperfunctions.

Definition 3.0.6. Recall that, according to Sato, the sheaf $\mathcal{B}_M$ of hyperfunctions on M is defined as

$$H^m(\mathrm{or}_M \otimes \mathrm{R}\Gamma_M(\mathcal{O}_X))$$

where $\mathrm{or}_M = H^m(\omega_M)$ is the orientation sheaf of M. So, it is natural to define the topological version $\mathcal{B}_M^\tau$ of $\mathcal{B}_M$ as

$$LH^m(\mathrm{or}_M^\tau \hat{\otimes} \mathrm{R}\Gamma_M(\mathcal{O}_X^\tau))$$

where $\mathrm{or}_M^\tau = LH^m(\omega_M^\tau)$ is the topological orientation sheaf of M. We also define a topological version $\mathcal{V}_M^\tau$ of the sheaf $\mathcal{V}_M$ of real analytic densities on M by setting

$$\mathcal{V}_M^\tau = \mathrm{or}_M^\tau \hat{\otimes} \Omega_{X|M}^\tau.$$

Proposition 3.0.7. *We have the following canonical isomorphisms :*

$$\mathrm{D}(\mathcal{A}_M^\tau) \simeq \mathcal{V}_M^\tau \hat{\otimes}_{\mathcal{A}_M^\tau} \mathcal{B}_M^\tau;$$

$$\mathrm{D}(\mathcal{V}_M^\tau) \simeq \mathcal{B}_M^\tau.$$

Proof. By part (b) of Proposition 3.0.3, we have

$$\mathrm{D}(\mathcal{A}_M^\tau) = \mathrm{R}\Gamma_M(\Omega_X^\tau)[m].$$

Hence,

$$\mathrm{D}(\mathcal{A}_M^\tau) \simeq \Omega_{X|M}^\tau \hat{\otimes}_{\mathcal{A}_M^\tau} \mathrm{R}\Gamma_M(\mathcal{O}_X^\tau)[m].$$

Using part (a) of Proposition 3.0.3 and the preceding definition, we have also

$$\mathrm{or}_M^\tau \hat{\otimes} \mathrm{R}\Gamma_M(\mathcal{O}_X^\tau)[m] \simeq \mathcal{B}_M^\tau.$$

Since

$$\mathrm{or}_M^\tau \hat{\otimes} \mathrm{or}_M^\tau \simeq \mathbb{C}_M^\tau,$$

we get

$$\mathrm{D}(\mathcal{A}_M^\tau) \simeq \Omega_{X|M}^\tau \hat{\otimes} \mathrm{or}_M^\tau \hat{\otimes} \mathcal{B}_M^\tau \simeq \mathcal{V}_M^\tau \hat{\otimes} \mathcal{B}_M^\tau.$$

Since $\mathcal{V}_M^\tau$ is locally isomorphic to $\mathcal{A}_M^\tau$, the second part of the result follows easily from the first one. □

4 Continous correspondences between $\mathcal{A}_M$ and $\mathcal{B}_N$

Proposition 4.0.8. *Let M, N be real analytic manifolds of dimension m, n and let*

$$p_M : M \times N \to M \qquad \text{and} \qquad p_N : M \times N \to N$$

be the canonical projections. Then,

$$\mathrm{R}\mathcal{L}\,(p_M^{-1}\mathcal{A}_M^\tau, p_N^! \mathcal{B}_N^\tau) \simeq p_M^{-1}\mathcal{V}_M^\tau \,\hat{\otimes}_{p_M^{-1}\mathcal{A}_M^\tau}\, \mathcal{B}_{M\times N}^\tau.$$

Proof. Let X, Y be Stein complexifications of M and N and let

$$i_M : M \to X, \quad i_N : N \to Y, \quad i_{M\times N} : M \times N \to X \times Y$$

be the canonical inclusions. Denote

$$p_X : X \times Y \to X, \quad p_Y : X \times Y \to Y$$

the canonical projections. Using a few well-known functorial isomorphisms of sheaf theory, we get

$$\begin{aligned} i_{M\times N}^! \mathrm{R}\mathcal{L}\,(p_X^{-1}\mathcal{O}_X^\tau, p_Y^! \mathcal{O}_Y^\tau) &\simeq \mathrm{R}\mathcal{L}\,(i_{M\times N}^{-1} p_X^{-1}\mathcal{O}_X^\tau, i_{M\times N}^! p_Y^! \mathcal{O}_Y^\tau) \\ &\simeq \mathrm{R}\mathcal{L}\,(p_M^{-1} i_M^{-1}\mathcal{O}_X^\tau, p_N^! i_N^! \mathcal{O}_Y^\tau). \end{aligned}$$

We know that

$$\mathrm{or}_N^\tau \hat{\otimes} \mathrm{R}\Gamma_N(\mathcal{O}_Y^\tau)[n] \simeq \mathcal{B}_N^\tau.$$

Therefore,

$$i_N^! \mathcal{O}_Y^\tau \simeq \mathrm{or}_N^\tau \hat{\otimes} \mathcal{B}_N^\tau[-n].$$

Hence,

$$\begin{aligned} i_{M\times N}^! \mathrm{R}\mathcal{L}\,(p_X^{-1}\mathcal{O}_X^\tau, p_Y^! \mathcal{O}_Y^\tau) &\simeq \mathrm{R}\mathcal{L}\,(p_M^{-1}\mathcal{A}_M^\tau, p_N^!(\mathrm{or}_N^\tau \hat{\otimes} \mathcal{B}_N^\tau[-n])) \\ &\simeq p_N^{-1}\,\mathrm{or}_N^\tau \hat{\otimes} \mathrm{R}\mathcal{L}\,(p_M^{-1}\mathcal{A}_M^\tau, p_N^! \mathcal{B}_N^\tau)[-n] \end{aligned} \tag{*}$$

Using the results recalled in Subsection 2.3, we have

$$\mathrm{R}\mathcal{L}\,(p_X^{-1}\mathcal{O}_X^\tau, p_Y^! \mathcal{O}_Y^\tau) \simeq \Omega_{X\times Y}^{(m,0)\tau}[m]$$

where the right hand term denotes the topological sheaf associated to the sheaf of holomorphic forms of degree m in the X variables and 0 in the Y variables. From the preceding isomorphism it follows that

$$\begin{aligned} i_{M\times N}^! \mathrm{R}\mathcal{L}\,(p_X^{-1}\mathcal{O}_X^\tau, p_Y^! \mathcal{O}_Y^\tau) &\simeq i_{M\times N}^! \Omega_{X\times Y}^{(m,0)\tau}[m] \\ &\simeq p_M^{-1}\Omega_{X|M}^\tau \,\hat{\otimes}_{p_M^{-1}\mathcal{A}_M^\tau}\, \mathrm{or}_{M\times N}^\tau \hat{\otimes} \mathcal{B}_{M\times N}^\tau[-n]. \end{aligned}$$

Since

$$\mathrm{or}_{M\times N}^\tau \simeq \mathrm{or}_M^\tau \,\hat{\boxtimes}\, \mathrm{or}_N^\tau,$$

we get

$$i_{M\times N}^! \mathrm{R}\mathcal{L}\,(p_X^{-1}\mathcal{O}_X^\tau, p_Y^! \mathcal{O}_Y^\tau) \simeq p_N^{-1}\,\mathrm{or}_N^\tau \hat{\otimes} p_M^{-1}\mathcal{V}_M^\tau \,\hat{\otimes}_{p_M^{-1}\mathcal{A}_M^\tau}\, \mathcal{B}_{M\times N}^\tau[-n].$$

Combining this isomorphism with (*) and using the fact that

$$\mathrm{or}_N^\tau \,\hat{\otimes}\, \mathrm{or}_N^\tau \simeq \mathbb{C}_N^\tau,$$

we get the isomorphism

$$\mathrm{R}\mathcal{L}\,(p_M^{-1}\mathcal{A}_M^\tau, p_N^! \mathcal{B}_N^\tau) \simeq p_M^{-1}\mathcal{V}_M^\tau \,\hat{\otimes}_{p_M^{-1}\mathcal{A}_M^\tau}\, \mathcal{B}_{M\times N}^\tau.$$

□

Corollary 4.0.9. *Let M, N be real analytic manifolds. Then,*

$$\mathrm{RL}\,(\mathrm{R}\Gamma_c(M;\mathcal{A}_M^\tau), \mathrm{R}\Gamma(N;\mathcal{B}_N^\tau)) \simeq \mathrm{R}\Gamma(M\times N; p_M^{-1}\mathcal{V}_M^\tau \,\hat{\otimes}_{p_M^{-1}\mathcal{A}_M^\tau}\, \mathcal{B}_{M\times N}^\tau)$$

and

$$\mathrm{RL}\,(\mathcal{A}_M^\tau, \mathcal{B}_M^\tau) \simeq \mathrm{R}\Gamma_\Delta(M\times M; p_M^{-1}\mathcal{V}_M^\tau \,\hat{\otimes}_{p_M^{-1}\mathcal{A}_M^\tau}\, \mathcal{B}_{M\times M}^\tau)$$

where Δ is the diagonal of M.

Proof. By well-known results of sheaf theory, for any topological sheaf F on M and any topological sheaf G on N, we have

$$\mathrm{R}\Gamma(M\times N; \mathrm{R}\mathcal{L}\,(p_M^{-1}F, p_N^! G)) \simeq \mathrm{RL}\,(\mathrm{R}\Gamma_c(M;F), \mathrm{R}\Gamma(N;G))$$

and, if $M=N$, we also have

$$\mathrm{R}\Gamma_\Delta(M\times M; \mathrm{R}\mathcal{L}\,(p_M^{-1}F, p_M^! F)) \simeq \mathrm{RL}\,(F,F).$$

By combining these isomorphisms with the preceding proposition, we get the conclusion. □

Remark 4.0.10. The first isomorphism of the preceding corollary means in particular that any continuous cohomological correspondence between $\mathcal{A}_M^\tau$ and $\mathcal{B}_N^\tau$ is realized by an integral transform whose kernel is a section of

$$p_M^{-1}\mathcal{V}_M \otimes_{p_N^{-1}\mathcal{A}_M} \mathcal{B}_{M\times N}.$$

This can be viewed as a real analytic version of the well-known kernels theorem of Schwartz. As for the second isomorphism, it means in particular that continuous local operators from $\mathcal{A}_M^\tau$ to $\mathcal{B}_M^\tau$ are represented by a hyperfunction kernel whose support is in the diagonal Δ of M.

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