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Second Hyperfunctions, Regular Sequences, and Fourier Inverse Transforms

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To the memory of our friend Pascal

Abstract Second hyperfunctions are formal boundary values of microfunctions with holomorphic parameters defined on wedges in much the same way in which classical hyperfunctions are boundary values of holomorphic functions defined on wedges. Since microfunctions with holomorphic parameters are themselves already defined in a formal way, second hyperfunctions have a rather non-intuitive definition and few explicit examples of second hyperfunctions which are not classical are known. In this paper we shall show that one can arrive at a better understanding by introducing the notion of regular sequences of holomorphic functions. We shall then show that representation of second hyperfunctions in terms of regular sequences is quite efficient in the context of regularization of the Fourier-inverse transform of functions which appear in second microlocalization.

Keywords: second hyperfunctions, microfunctions with holomorphic parameters, Fourier-transform

1 Introduction

Second hyperfunctions, are the natural frame for second microlocalization. They have been introduced in 1970 by T.Kawai and M.Kashiwara, as a very natural extension of standard hyperfunctions and have many interesting properties. In particular they form a rather large space of generalized functions in which calculations are often easier to perform than in classical hyperfunctions or in distributions. Related to this is the fact that it may happen that certain calculations are meaningless in classical hyperfunctions whereas they have an easy interpretation in second hyperfunctions: a classical example is the function $1/(x_2 - ix_1^2 + i0)$, which can be given no meaning in classical hyperfunctions, but which has an easy interpretation in second hyperfunctions and gives a fundamental solution for Mizohata's operator $p(x, D) = (\partial/\partial x_1) + ix_1(\partial/\partial x_2)$. (See [4].) Few other explicit second hyperfunctions which are not classical hyperfunctions are known however at present.

The basic ingredient in the definition of second hyperfunctions is given by so-called "microfunctions with holomorphic parameters". Such microfunctions play in higher microlocalization roughly speaking the role of holomorphic functions with hyperfunctional values. Indeed, second hyperfunctions are formal boundary values of microfunctions with holomorphic parameters defined on wedges in much the same way in which classical hyperfunctions are boundary values of holomorphic functions defined on wedges. However, microfunctions with holomorphic parameters themselves are defined in terms of cohomology classes, so, all in all, the definition of second hyperfunctions becomes highly non-intuitive. A specific difficulty is, that despite the fact that second hyperfunctions are objects living on the real space, the holomorphic functions which are ultimately involved in their definition, have domains of definition which are somewhat distant from the real space and which are often quite small. On the other hand, in explicit calculations it would often be convenient to use these holomorphic functions to represent cohomology classes, and then it would be desirable to dispose of large domains of definition, which come as close as possible to the real space.

In the described context the goal which we have set ourselves for this paper is twofold. On one hand, we shall show that it is possible to define microfunctions with holomorphic parameters of the type needed in second microlocalization by sequences of holomorphic functions which have domains of definition which become increasingly large and which exhaust the wedge-type domains on which we would like our defining functions to live. In particular, on any previously fixed compact set inside such a wedge, one will be able to work with just one single defining function and the domains of definition of these holomorphic functions will come arbitrarily close to the real space if the function in the sequence is chosen appropriately. The basic new notion is here that of a regular sequence of holomorphic functions $\{h_j\}_{j\geq 1}$, so actually second microfunctions will be defined in the end by sequences of holomorphic functions defined on on an increasing sequence of wedges, rather than by single holomorphic functions defined on a fixed wedge. All this is roughly speaking the content of the sections 2 to 4.

Related to this, is our second goal, which is part of an ongoing attempt of the authors to construct a theory of the local Fourier transform for second hyperfunctions. The basic observation is here that second hyperfunctions appear often in the form of formal Fourier-integrals with non-integrable integrands, and the first thing to do is to give a regularization procedure for these integrals. We shall do this, after some preparations, in section 7 and it will turn out, that the best way to argue, is precisely to use the notion of regular sequences of holomorphic functions which we mentioned just before. In section 9 we shall show how one can use this theory to construct fundamental solutions for operators of a certain form. Since some of these operators are known to not admit fundamental solutions in classical hyperfunctions, we obtain in this way in particular examples of second hyperfunctions which are not classical. (See however also [8]).

We dedicate this paper to Pascal Laubin, who was one of the main contributors to the theory of second microlocalization. In particular, we would like to cite his recent paper [9], of which many implications should still be worked out.

2 Definitions and boundary value representations

In this section we discuss boundary value representations for microfunctions with holomorphic parameters and for second microfunctions in local coordinates. This is necessary on one hand to review the main definitions and on the other hand since for the second part of the section the results which can be found in the literture are not explicit enough for our needs. We also refer to [15], [6], [5], [14] and [7].

We start from the sequence of inclusions

$$\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d} \to \mathbb{R}^d \times \mathbb{C}^{n-d} \to \mathbb{C}^d \times \mathbb{C}^{n-d} = \mathbb{C}^n,$$

where each map is constructed from the standard complexification $\mathbb{R}^k \to \mathbb{C}^k$. We denote the three manifolds $\mathbb{R}^n$, $\mathbb{R}^d \times \mathbb{C}^{n-d}$ and $\mathbb{C}^n$ by M, N and X respectively. The second inclusion $N \to X$ defines the conormal bundle

$$\tilde{\Sigma} := T^*_N X \simeq T^*_{\mathbb{R}^d}\mathbb{C}^d \times \mathbb{C}^{n-d} \overset{\pi_N}{\to} N$$

along N, and the inclusion $M \to X$ defines the conormal bundle

$$T^*_M X \overset{\pi_M}{\to} M$$

along M. We define the real regular involutive submanifold

$$\Sigma := \tilde{\Sigma} \times_{T^*X} T^*_M X,$$

in $T^*_M X$. $\tilde{\Sigma}$ is identified with the union of the bicharacteristics of the complexification $\Sigma^{\mathbb{C}}$ of Σ which pass through points in Σ. We also consider the bundle $T^*_\Sigma\tilde{\Sigma} \to \Sigma$ associated with $\Sigma \to \tilde{\Sigma}$.

We take coordinate systems $x = (x_1, \dots, x_n) = (x', x'')$, $z = x + iy = (z_1, \dots, z_n) = (z', z'')$, and (x', z'') of M, X, and N respectively, with the conventions $x' = (x_1, \dots, x_d)$, $x'' = (x_{d+1}, \dots, x_n)$, $z' = (z_1, \dots, z_d)$ and $z'' = (z_{d+1}, \dots, z_n)$. Coordinate systems of $\tilde{\Sigma}$, Σ, and $T^*_M X$ are written as $(x', z''; \xi')$, $(x; \xi')$, and $(x; \xi)$ respectively, where $\xi = (\xi_1, \dots, \xi_n) = (\xi', \xi'')$ denotes the variables dual to x and ξ' and ξ'' denote $(\xi_1, \dots, \xi_d)$ and $(\xi_{d+1}, \dots, \xi_n)$.

We introduce the sheaf $\mathcal{CO}_N$ of microfunctions with holomorphic parameters on $\tilde{\Sigma} = T^*_N X$ by

$$\mathcal{CO}_N := \mu_N(\mathcal{O}_X) \otimes or_{N/X}[d]$$

and the sheaf $\mathcal{BO}_N$ of hyperfunctions with holomorphic parameters on N by

$$\mathcal{BO}_N := \mathcal{CO}_N|_N.$$

Here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X, μ_N Sato's microlocalization functor along N, and $or_{N/X}$ the relative orientation sheaf. These sheaves, together with the sheaf

$$\mathcal{AO}_N := \mathcal{O}_X|_N,$$

form an exact sequence

$$0 \to \mathcal{AO}_N \to \mathcal{BO}_N \overset{\mathrm{sp}_N}{\to} \dot{\pi}_{N,*}\mathcal{CO}_N \to 0, \tag{2.1}$$

where $\dot{\pi}_N$ denotes the projection $\dot{T}^*_N X = T^*_N X \setminus N \to N$.

We also introduce the sheaf $\mathcal{C}^2_\Sigma$ of second microfunctions along Σ on $T^*_\Sigma \tilde{\Sigma}$ and the sheaf $\mathcal{B}^2_\Sigma$ of second hyperfunctions along Σ on Σ by

$$\mathcal{C}^2_\Sigma := \mu_\Sigma(\mathcal{CO}_N) \otimes or_{\Sigma/\tilde{\Sigma}}[n-d],$$
$$\mathcal{B}^2_\Sigma := \mathcal{C}^2_\Sigma|_\Sigma.$$

Denoting by

$$\mathcal{A}^2_\Sigma := \mathcal{AO}_N|_\Sigma,$$

we thus obtain the exact sequences

$$0 \to \mathcal{A}^2_\Sigma \to \mathcal{B}^2_\Sigma \overset{\mathrm{sp}^2_\Sigma}{\to} \dot{\pi}_{\Sigma,*}\mathcal{C}^2_\Sigma \to 0, \tag{2.2}$$
$$0 \to \mathcal{C}_M|_\Sigma \to \mathcal{B}^2_\Sigma, \tag{2.3}$$

where $\dot{\pi}_\Sigma$ denotes the projection $\dot{T}^*_\Sigma \tilde{\Sigma} = T^*_\Sigma \tilde{\Sigma} \setminus \Sigma \to \Sigma$. We say that a second hyperfunction u is classical if it belongs to the image of $\mathcal{C}_M$ in (2.3).

Having defined microfunctions with holomorphic parameters and second hyperfunctions in a rather formal way, we now want to turn to the problem of boundary value representations.

The first remark is that, locally, a hyperfunction $v(x', z'')$ with holomorphic parameter z'' is a class of a finite formal sum of holomorphic functions

$$v = \Big[\sum_j F_j\Big], \quad F_j \in \mathcal{O}((U' + iG'_j) \times V'' \cap \{|\operatorname{Im} z'| < \delta\}), \tag{2.4}$$

with some positive constant δ, open subsets $U' \subset \mathbb{R}^d$ and $V'' \subset \mathbb{C}^{n-d}$, and a family $\{G'_j\}_j$ of open convex cones in $\mathbb{R}^d$. The right hand side is zero in a neighborhood of a point $(\dot{x}', \dot{z}'') \in U' \times V''$ if there exist a positive $\tilde{\delta}$, a neighborhood $\tilde{U}' \times \tilde{V}''$ of $(\dot{x}', \dot{z}'')$, a family of open convex cones $\{\tilde{G}'_j\}_j$ with $\tilde{G}'_j \subset G'_j$, and a family of holomorphic functions $\{F_{jk}\}_{j,k}$ such that each F_{jk} is holomorphic on the set

$$(\tilde{U}' + i(\tilde{G}'_j + \tilde{G}'_k)) \times \tilde{V}'' \cap \{|\operatorname{Im} z'| < \tilde{\delta}\}$$

and that satisfies $F_{jk} = -F_{kj}$ for any j and k, and $F_j = \sum_k F_{jk}$ for any j.

A class represented by a single holomorphic function F is denoted by $b_{\operatorname{Im} z' \to 0}(F)$, or simply by $b(F)$. Thus we can write

$$v = \sum_j b_{\operatorname{Im} z' \to 0}(F_j).$$

A microfunction $\operatorname{sp}_N(v)$ with holomorphic parameter is a class represented by a hyperfunction v with holomorphic parameter. $\operatorname{sp}_N(v)$ is zero in a neighborhood of a point $\dot{q} = (\dot{x}', \dot{z}''; \dot{\xi}')$ if v has a representation (2.4) in a neighborhood of $(\dot{x}', \dot{z}'')$ such that $\dot{\xi}' \notin G'^{\perp}_j$ for any j. The microfunction represented by the boundary value of a single holomorphic function F is also denoted by $b_{\operatorname{Im} z' \to 0}(F)$, or simply by $b(F)$, if there is no risk of confusion. It is well-known that every microfunction v with holomorphic parameter admits, locally, a representation

$$v = b_{\operatorname{Im} z' \to 0}(F).$$

A second hyperfunction $u(x', x'')$ is, locally, a class of a finite formal sum of microfunctions with holomorphic parameters

$$u = \Big[\sum_j v_j\Big], \quad v_j \in \mathcal{CO}(U' \times (U'' + iG''_j) \times \Gamma' \cap \{|\operatorname{Im} z''| < \delta\}), \tag{2.5}$$

with some positive δ, open subsets $U' \subset \mathbb{R}^d$ and $U'' \subset \mathbb{R}^{n-d}$, an open cone $\Gamma' \subset \mathbb{R}^d$, and a family $\{G''_j\}_j$ of open convex cones in $\mathbb{R}^{n-d}$. The right hand side is zero in a neighborhood of a point $\dot{q} = (\dot{x}', \dot{x}''; \dot{\xi}')$ if there exist a positive $\tilde{\delta}$, a neighborhood $\tilde{U}' \times \tilde{U}'' \times \tilde{\Gamma}'$ of $\dot{q}$, a family of open convex cones $\{\tilde{G}''_j\}_j$ with $\tilde{G}''_j \subset G''_j$, and a finite family of microfunctions with holomorphic parameter $\{v_{jk}\}_{j,k}$, such that each v_{jk} is defined on

$$(\tilde{U}' \times (\tilde{U}'' + i(\tilde{G}''_j + \tilde{G}''_k)) \times \tilde{\Gamma}') \cap \{|\operatorname{Im} z''| < \tilde{\delta}\},$$

such that $v_{jk} = -v_{kj}$ for any j and k, and $v_j = \sum_k v_{jk}$ for any j.

A class represented by a single microfunction v is denoted by $b_{\operatorname{Im} z'' \to 0}(v)$, or, simply by $b(v)$. Thus we can write

$$u = \sum_j b_{\operatorname{Im} z'' \to 0}(v_j).$$

A second microfunction $\operatorname{sp}^2_\Sigma(u)$ is a class represented by a second hyperfunction u, and $\operatorname{sp}^2_\Sigma(u) = 0$ in a neighborhood of a point $\dot{q} = (\dot{x}; \dot{\xi}'; \dot{\xi}'')$ if u has a representation (2.5) in a neighborhood of $(\dot{x}; \dot{\xi}')$ such that $\dot{\xi}'' \notin G''^{\perp}_j$ for any j.

It is useful to understand the imbedding morphism (2.3) in terms of boundary value representations, and we shall now explain how to achieve this.

We recall at first that a microfunction $u \in \mathcal{C}_M$ defined in a neighborhood of a point $\dot{q} = (\dot{x}; \dot{\xi}') \in \Sigma$ can be written in the form

$$u = b_{\operatorname{Im} z \to 0}(F),$$

where F is a holomorphic function on $\{z = x + iy \in U + iG; |y| < \delta\}$ with an open neighborhood $U \subset \mathbb{R}^n$ of $\dot{x}$, an open convex cone $G \subset \mathbb{R}^n$ and a positive δ. For this defining function, we take an open convex neighborhood $\tilde{U} \subset U$ of $\dot{x}$, an open convex subcone $\tilde{G} \subset G$, a positive $\tilde{\delta} \le \delta$, a finite family $\{G''_j\}_j$ of open convex cones in $\mathbb{R}^{n-d}$, and a family $\{F_j\}_j$ of holomorphic functions such that each F_j is holomorphic on the set

$$\{z = x + iy \in \tilde{U} + i(G + G''_j); |y| < \tilde{\delta}\},$$

and that F is decomposed as

$$F(z) = \sum_j F_j(z), \tag{2.6}$$

on their common domain of definition. Here we have used the convention

$$G + G''_j = \{(y', y'' + \tilde{y}'') \in \mathbb{R}^n; (y', y'') \in G, \tilde{y}'' \in G''_j\}.$$

Each F_j defines a microfunction with holomorphic parameters $v_j = b_{\operatorname{Im} z' \to 0}(F_j)$ on

$$\{(x', z'' = x'' + iy''; \xi'); (x', x'') \in \tilde{U}, y'' \in G''_j, |y''| < \tilde{\delta}, \xi' \in \mathbb{R}^d\},$$

and the family $\{v_j\}_j$ defines a second microfunction $\tilde{u}$ in a neighborhood of $\dot{q}$. The correspondence $u \mapsto \tilde{u}$ gives precisely the imbedding morphism (2.3).

Remark 2.1. *In the explanation above, we have omitted the details concerning the existence of the functions $\{F_j\}_j$ and the ones related to the fact that $\tilde{u}$ does not depend on the choice of F, $\{G''_j\}_j$ and $\{F_j\}_j$. We only note here that for any choice of $\{G''_j\}_j$ satisfying $\bigcup_j G''^{\perp}_j = \dot{\mathbb{R}}^{n-d}$, the existence of a decomposition as in (2.6) can be shown using the vanishing theorem of cohomology groups of convex sets in $\mathbb{C}^n$ with $\mathcal{O}$-coefficients, and that the "independency" is proved with the aid of some kind of edge of the wedge theorem.*

We now come to the main result in this section.

Theorem 2.2. *Consider pairs of open sets $\tilde{U}' \subset\subset U'$ in $\mathbb{R}^d$ and $\tilde{V}'' \subset\subset V''$ in $\mathbb{C}^{n-d}$. Also consider an open proper convex cone $\Gamma' \subset \dot{\mathbb{R}}^d$, some strict subcone $\tilde{\Gamma}'$ of Γ', and an open convex cone $G' \subset \dot{\mathbb{R}}^d$ with $G' \subset \Gamma'^{\perp}$. Assume that the closure of $\tilde{V}''$ is polynomially convex. Then for any $u \in \mathcal{CO}_N(U' \times V'' \times \Gamma')$, there exists a holomorphic function $F \in \mathcal{O}((\tilde{U}' + iG') \times \tilde{V}'')$ such that $u = b(F)$ in $\mathcal{CO}_N(\tilde{U}' \times \tilde{V}'' \times \tilde{\Gamma}')$.*

Recall that a compact set in $\mathbb{C}^m$ is called polynomially convex if it is an intersection -finite or not- of sets of form $\{z \in \mathbb{C}^m; |f(z)| \leq 1\}$ where f is a polynomial. Also recall that an intersection of sets of form $\{z \in \mathbb{C}^m; \operatorname{Re} f(z) \geq 0\}$ with some polynomial f is polynomially convex if it is compact. In fact, for such a set K and a point $\dot{z} \notin K$, there exists a polynomial f with $K \subset \{\operatorname{Re} f \geq 0\}$ and $\operatorname{Re} f(\dot{z}) < 0$. By taking $r = \max_{z \in K} |f(z)|$ and $g_s(z) = (f(z) - s)^2/(r^2 + s^2)$, we have

$$K \subset \bigcap_{s>0} \{z; |g_s(z)| \leq 1\} = \{z; |f(z)| \leq r, \operatorname{Re} f(z) \geq 0\}.$$

Thus, in particular, any convex compact set is polynomially convex.

Corollary 2.3. *Let U', V'', Γ' and $u \in \mathcal{CO}_N(U' \times V'' \times \Gamma')$ be as in theorem 2.2. Also consider a sequence of open sets $U'_j \subset\subset U'$, of open sets $V''_j \subset\subset V''$, and of open convex proper cones $G'_j \subset\subset \Gamma'^{\perp}$ such that $U'_j \subset U'_{j+1}$, $V''_j \subset V''_{j+1}$, $G'_j \subset G'_{j+1}$ and*

$$\bigcup_{j=1}^{\infty} U'_j = U', \quad \bigcup_{j=1}^{\infty} V''_j = V'', \quad \bigcup_{j=1}^{\infty} G'_j = \operatorname{Int} \Gamma'^{\perp}.$$

Assume that the closure of each V''_j is polynomially convex. Then there exists a sequence of holomorphic functions $F_j \in \mathcal{O}((U'_j + iG'_j) \times V''_j)$ such that for every j, $u = b(F_j)$ on $U'_j \times V''_j \times \Gamma'_j$.

Before we start the proof we recall the following

Proposition 2.4. *Let $K = K' \times K''$ be a compact set in $\mathbb{R}^d \times \mathbb{C}^{n-d}$, where K' is a compact set in $\mathbb{R}^d$ and K'' is a polynomially convex compact set in $\mathbb{C}^{n-d}$. Then*

$$H^j(K, \mathcal{AO}) = 0, \quad \forall j = 1, 2, \ldots. \tag{2.7}$$

Here $H^j(K, \mathcal{AO})$ denotes the j-th cohomology group of K with coefficients in $\mathcal{AO}$. Related notations used later on should be self-explanatory.

(To prove proposition 2.4, it suffices to observe that K has a fundamental system of neighborhoods in $\mathbb{C}^n$ which are Stein, and that for any sheaf $\mathcal{F}$ on $\mathbb{C}^n$

$$H^j(K, \mathcal{F}|_{\mathbb{R}^d \times \mathbb{C}^{n-d}}) = \varinjlim_{U \supset\supset K} H^j(U, \mathcal{F}).$$

Cf. e.g. [2]. That the K in the statement admits a fundamental system of Stein neighborhoods follows from the following two remarks: any compact set K' in $\mathbb{R}^d$ admits a fundamental system of neighborhoods in $\mathbb{C}^d$ which are Stein, and any polynomially convex set in $\mathbb{C}^{n-d}$ admits a fundamental systems of neighborhoods in $\mathbb{C}^{n-d}$ which are Stein.)

Applying proposition 2.4 to the exact sequence (2.1), we obtain the following corollary

Corollary 2.5. *Let $K = K' \times K''$ be as above. Then the map*

$$\mathcal{BO}(K) \to \mathcal{CO}(\dot{\pi}^{-1}(K)) \tag{2.8}$$

is surjective.

We also need:

Proposition 2.6. *Let $\tilde{U}' \subset\subset U'$ and $\tilde{V}'' \subset\subset V''$ be as in theorem 2.2, and $\Gamma' \subset \dot{\mathbb{R}}^d$ a proper open convex cone. Consider $u \in \mathcal{BO}(U' \times V'')$ such that*

$$\mathrm{WF}_A(u) \subset U' \times V'' \times \Gamma'. \tag{2.9}$$

Also fix some open convex cone $G' \subset\subset \Gamma'^{\perp}$. Then there exist $\delta > 0$ and $F \in \mathcal{O}((\tilde{U}' + iG') \times \tilde{V}'' \cap \{|\operatorname{Im} z'| < \delta\})$ such that $u = b(F)$ on $\tilde{U}' \times \tilde{V}''$.

We prepare the proof of proposition 2.6 with

Lemma 2.7. *For a closed set $K' \subset \mathbb{R}^d$ and a closed cone $\Gamma' \subset \dot{\mathbb{R}}^d$, there exists a sheaf morphism*

$$\Phi = \Phi_{K' \times \Gamma'} : \mathcal{CO}_N|_{\dot{T}^*_N X} \to \mathcal{CO}_N|_{\dot{T}^*_N X}$$

such that $\Phi = \mathrm{id}$ on $\operatorname{Int} K' \times \mathbb{C}^{n-d} \times \operatorname{Int} \Gamma'$ and $\Phi = 0$ outside $K' \times \mathbb{C}^{n-d} \times \Gamma'$.

Proof. Let δ be the Delta-Dirac distribution at $0 \in \mathbb{R}^d$. We take a microfunction $k(x', y') \in \mathcal{C}_{\mathbb{R}^{2d}}(\dot{T}^*_{\mathbb{R}^{2d}}\mathbb{C}^{2d})$ satisfying

$$k(x', y') = \delta(x' - y') \text{ on } \{(x', x'; \xi', -\xi'); (x', \xi') \in \operatorname{Int}(K' \times \Gamma')\},$$
$$\operatorname{supp} k \subset \{(x', x'; \xi', -\xi'); (x', \xi') \in K' \times \Gamma'\}.$$

The existence of such microfunctions is clear from the conical flabbiness of the sheaf of microfunctions. (Here we observe that the support of $\delta(x' - y')$ as a microfunction is $\{(x', y'; \xi', \eta'); x' = y', \xi' + \eta' = 0\}$.)

To conclude the proof we define $\Phi = \Phi_{K' \times \Gamma'}$ by

$$\Phi(u)(x', z'') = \int k(x', y') u(y', z'') dy'.$$

It follows from the support property $\operatorname{supp} k \subset \{x' = y', \xi' + \eta' = 0\}$ that Φ becomes a sheaf morphism of $\mathcal{CO}_N$ outside the zero section. The remaining statement also follows from the two properties of k. □

Proof of proposition 2.6. Consider a pair of compact sets $K' \subset\subset L'$ in $\mathbb{R}^d$ with $\tilde{U}' \subset\subset K' \subset\subset U'$, and an open set W'' in $\mathbb{C}^{n-d}$ with $\tilde{V}'' \subset\subset W'' \subset\subset V''$.

If we can find some hyperfunction with holomorphic parameters $w \in \mathcal{BO}_N(\mathbb{R}^d \times W'')$ with the property that $u - w$ is analytic in a neighborhood of $K' \times W''$ and such that $\operatorname{supp} w \subset L' \times W''$, then we can obtain a defining function F of w by curvilinear Radon transformation of w with respect to the x' variables. What remains to be done will then be to adjust F so as to have $u = b(F)$. This can then be done by adding to F the analytic function $u - w$.

We will now explain how to obtain K', L', W'' and w with the above properties.

We take a compact neighborhood K' of $\tilde{U}'$ in U'. Applying to u, the sheaf morphism Φ as in lemma 2.7 for $K' \times \dot{\mathbb{R}}^d$, we obtain

$$\Phi(\operatorname{sp}_N(u)) \in \mathcal{CO}_N(U' \times V'' \times \dot{\mathbb{R}}^d).$$

Since this is 0 for x' outside K', we can extend it by zero to obtain $v \in \mathcal{CO}_N(\mathbb{R}^d \times V'' \times \dot{\mathbb{R}}^d)$, which satisfies $\operatorname{supp} v \subset K' \times V'' \times \Gamma'$ and $v = \operatorname{sp}_N(u)$ on $\{x' \in \operatorname{Int} K'\}$.

We also take a compact subset $L' \subset \mathbb{R}^d$ with $K' \subset\subset L'$ with analytic boundary, and a polynomially convex neighborhood K'' of $\tilde{V}''$ in V''. Applying corollary 2.5 to $L' \times K''$, we get $\tilde{u} \in \mathcal{BO}_N(L' \times K'')$ for which $v = \operatorname{sp}_N(\tilde{u})$ holds on a neighborhood of $\dot{\pi}^{-1}(L' \times K'')$. Since $\tilde{u}$ is analytic in a neighborhood of $\{x' \in \partial L'\}$, we can multiply $\tilde{u}$ with the characteristic function $\chi_{L'}(x')$ of L' and get

$$w = \chi_{L'}(x')\tilde{u}(x', z'') \in \mathcal{BO}_N(\mathbb{R}^d \times W''),$$

where $W'' = \operatorname{Int} K''$. This w, together with K', L' and W'' satisfies the desired properties. □

Remark 2.8. *It is not difficult to show that in proposition 2.6 one can always take "$\delta = \infty$" if one replaces the condition "$u = b(F)$" by the condition* $\operatorname{WF}_A(u - b(F)) = \emptyset$.

Proof of theorem 2.2. By cutting off the support of u using lemma 2.7 and taking the zero extension, we may assume, from the beginning, that u is defined on $\mathbb{R}^d \times V'' \times \dot{\mathbb{R}}^d$ and that the support of u is included in the closure of $\tilde{U}' \times V'' \times \tilde{\Gamma}'$. Then we can use corollary 2.5 and take $\tilde{u} \in \mathcal{BO}_N(W)$ with $u = \operatorname{sp}_N(\tilde{u})$, where W is some neighborhood of $\tilde{U}' \times \tilde{V}''$. The function F will be the one associated by proposition 2.6 with $\tilde{u}$. □

3 Existence of suitable polynomially convex subsets

In this section we prove the existence of certain polynomially convex compact sets which will be needed later on. For notational conveniene, we work on $\mathbb{C}^m$ rather than on $\mathbb{C}^{n-d}$ and take $z = (z_1, \ldots, z_m) = x + iy$ as a coordinate system.

We denote by $B(r)$ the open ball in $\mathbb{R}^m$ of radius r, centered at the origin. We also use the following conventions: $G(r) = G \cap B(r)$ and $G(r, s) = G \cap \{y \in \mathbb{R}^m; r < |y| < s\}$, whenever we are given some open cone $G \subset \mathbb{R}^m$.

Proposition 3.1. *Let $K \subset\subset \tilde{K} \subset\subset \tilde{U} \subset\subset U$ be four subsets in $\mathbb{R}^m$ where K and $\tilde{K}$ are compact and $\tilde{U}$ and U are open and convex. Also consider two proper open convex cones $\tilde{G} \subset\subset G$ in $\dot{\mathbb{R}}^m$ and two positive constants $d_1 < d_2$. Then there exists a positive constant δ for which the following condition holds: for any positive ε we can find a polynomially convex compact subset $L \subset \mathbb{C}^m$ such that*

$$[\tilde{U} + i\tilde{G}(\varepsilon, d_1)] \cup [(\tilde{U} \setminus \tilde{K}) + i(\tilde{G}(d_1) \cup B(\delta))] \subset L \subset [U + iG(d_2)] \cup [(U \setminus K) + iB(d_2)].$$

Proof. L will be constructed in the form

$$L := (\overline{\tilde{U}} + i\{|y| \leq d_1\}) \cap \bigcap_{t \in K} \bigcap_{k=1}^{\ell} \{z \in \mathbb{C}^m; \operatorname{Re} f_{\sigma,r,\xi_k}(z - t) \geq 0\}$$

for some $\sigma > 0$, $r > 0$, $\xi_k \in \dot{\mathbb{R}}^m$ with $|\xi_k| = 1$ $(k = 1, \ldots, \ell)$, where $f_{\sigma,r,\xi}$ is the polynomial

$$f_{\sigma,r,\xi}(z) = -i\langle z, \xi\rangle + \sigma(z^2 - r^2)/2.$$

First, we choose finitely many vectors $\{\xi_k\}_{k=1,\ldots,\ell} \subset \mathbb{R}^m$ with $|\xi_k| = 1$ for which the cone

$$\hat{G} := \{y \in \mathbb{R}^m; \langle y, \xi_k\rangle > 0 \text{ for } k = 1, \ldots, \ell\}$$

satisfies $\tilde{G} \subset\subset \hat{G} \subset\subset G$.

Next we choose σ.

Lemma 3.2. *Let $\tilde{G} \subset\subset \hat{G}$ and $\{\xi_k\}_{k=1,\ldots,\ell}$ be as above. Define the sets $V_{\sigma,r}(\xi)$ by*

$$V_{\sigma,r}(\xi) := \{y \in \mathbb{R}^m; \operatorname{Re} f_{\sigma,r,\xi}(iy) \geq 0\} \text{ for } \xi \in \mathbb{R}^m,$$

$$V_{\sigma,r}(\xi_1, \ldots, \xi_\ell) := \bigcap_{k=1}^{\ell} V_{\sigma,r}(\xi_k).$$

Then, we have the following properties.

1. *For any $\sigma > 0$ and $r > 0$ it follows that $V_{\sigma,r}(\xi_1, \ldots, \xi_\ell) \subset\subset \hat{G}$.*

2. *For any $d_1 > 0$, we can find a positive constant $\sigma(d_1) > 0$ such that*

$$\tilde{G} \cap \{|y| \le d_1\} \subset V_{\sigma,0}(\xi_1, \dots, \xi_\ell), \text{ and } \tilde{G} \cap \{|y| = d_1\} \subset\subset V_{\sigma,0}(\xi_1, \dots, \xi_\ell)$$

hold whenever $0 \le \sigma < \sigma(d_1)$.

3. *Finally, let us consider some arbitrary ε with $0 < \varepsilon < d_1$. There is then a positive constant $r(\varepsilon, d_1) > 0$ for which*

$$\tilde{G} \cap \{\varepsilon \le |y| \le d_1\} \subset\subset V_{\sigma,r}(\xi_1, \dots, \xi_\ell)$$

holds whenever $0 \le \sigma < \sigma(d_1)$ and $0 \le r < r(\varepsilon, d_1)$.

Proof. 1. $V_{\sigma,r}(\xi)$ is compact if σ is positive. Moreover if r is also positive,

$$V_{\sigma,r}(\xi) = \{\langle y, \xi\rangle \ge \sigma(y^2 + r^2)/2\} \subset \{\langle y, \xi\rangle > 0\}.$$

2. Note that $V_{\sigma,r}(\xi)$ decreases as σ increases, that the set $\tilde{G} \cap \{|y| \le d_1\}$ is included in the convex hull of $\{0\} \cup (\tilde{G} \cap \{|y| = d_1\})$, and that the set $V_{\sigma,0}(\xi_1, \dots, \xi_\ell)$ is convex and always contains 0. Thus it suffices to show that for some $\sigma(d_1)$,

$$\tilde{G} \cap \{|y| = d_1\} \subset\subset V_{\sigma(d_1),0}(\xi_k) = \{2\langle y, \xi_k\rangle / y^2 \ge \sigma\}, \text{ for any } k.$$

Also note that since $\tilde{G} \subset\subset \hat{G}$ as cones, $\tilde{G} \cap \{|y| = d_1\} \subset\subset \hat{G}$ and that $\langle y, \xi_k\rangle$ is positive on $\hat{G}$. Thus $2\langle y, \xi_k\rangle / y^2$ has a positive lower bound for $y \in \tilde{G} \cap \{|y| = d_1\}$ and $k = 1, \dots, \ell$.

3. We can take such a $r(\varepsilon, d_1)$ using the facts that $V_{\sigma,r}(\xi)$ also decreases as r increases, that $\tilde{G} \cap \{\varepsilon \le |y| \le d_1\} \subset\subset V_{\sigma(d_1),0}(\xi_1, \dots, \xi_\ell)$, and that $2\langle y, \xi_k\rangle/\sigma - y^2 > 0$ on $V_{\sigma(d_1),0}(\xi_1, \dots, \xi_\ell)$. □

Lemma 3.3. *Let $\{\xi_k\}_{k=1,\dots,\ell}$ be as above. Define the sets $W_{\sigma,r}(t)$ by*

$$W_{\sigma,r}(t) := \{z = x + iy \in \mathbb{C}^m; \operatorname{Re} f_{\sigma,r,\xi_k}(z - t) \ge 0 \text{ for } k = 1, \dots, \ell\},$$

$$W_{\sigma,r,K} := \bigcap_{t \in K} W_{\sigma,r}(t).$$

Then we have the following properties:

1. *For any $\sigma > 0$ and any $r > 0$, we have an inclusion*

$$W_{\sigma,r,K} \supset \mathbb{R}^m + iV_{\sigma,r}(\xi_1, \dots, \xi_\ell),$$

and an equality

$$\{x + iy \in \mathbb{C}^m; x \in K\} \cap W_{\sigma,r,K} = K + iV_{\sigma,r}(\xi_1, \dots, \xi_\ell).$$

2. *We have the two inclusions*

$$\{x+iy \in \mathbb{C}^m; \operatorname{dist}(x,K) > r, y \in V_{\sigma,0}(\xi_1,\dots,\xi_\ell)\} \subset W_{\sigma,r,K},$$

and

$$\{x+iy \in \mathbb{C}^m; \operatorname{dist}(x,K) > r, |y| \le \sqrt{1/\sigma^2 - r^2 + \operatorname{dist}(x,K)^2} - 1/\sigma\} \subset W_{\sigma,r,K}.$$

Now we continue the proof of Proposition 3.1.

We fix σ with $\sigma < \sigma(d_1)$ as in lemma 3.2 and take r_0 with $2r_0 < \operatorname{dist}(K, \mathbb{R}^m \setminus \tilde{K})$ and $\delta = \sqrt{1/\sigma^2 + 3r_0^2} - 1/\sigma$. Then we have for $r < r_0$,

$$\{x+iy; x \notin \tilde{K}, |y| < \delta\} \subset W_{\sigma,r,K}.$$

For any $\varepsilon > 0$, it is enough to take r with $0 < r < r(\varepsilon, d_1)$ and $r < r_0$ for which

$$L := (\overline{\tilde{U}} + i\{|y| \le d_1\}) \cap W_{\sigma,r,K}$$

satisfies the condition in the statement. □

4 Regular sequences of holomorphic functions

As we saw in section 2, a microfunction $v(x', z'')$ with holomorphic parameters z'' is, locally in the z'' variables, represented as a boundary value for $\operatorname{Im} z' \to 0$ of a holomorphic function $h(z', z'')$. However, in general, it will not be possible to obtain such representations globally. In order to use microfunctions with holomorphic parameters as defining functions for second microfunctions, we shall now represent a global section $v(x', z'')$ of microfunctions with holomorphic parameters in the form of a "boundary value of a sequence of holomorphic functions". On overlapping parts of the domains of definition, the single holomorphic functions will define the same microfunction v and the domains of definition of these functions with respect to the z'' variables will be increasing in such a way that they will ultimatively exhaust the domain of definition of v.

To state our main definition, consider an open set $U' \subset \mathbb{R}^d$, an open convex cone $G' \subset \dot{\mathbb{R}}^d$, and an open set $V'' \subset \mathbb{C}^{n-d}$. Also fix $\Gamma' \subset \dot{\mathbb{R}}^d$.

Definition 4.1. *A sequence $\{h_j(z)\}_{j=1,2,\dots}$ of holomorphic functions is called a regular sequence of holomorphic functions on $U' \times V'' \times \Gamma'$ if the following happens:*

a) there exists an increasing sequence of open subsets $\{V''_j\}_{j=1,2,\dots}$ in V'' which exausts V'', in the sense that $\bigcup_j V''_j = V''$, and a sequence of positive numbers $\{\delta_j\}_{j=1,2,\dots}$, such that the function h_j is defined on the set

$$\{z = (z', z'') \in (U' + iG') \times V''_j; |\operatorname{Im} z'| < \delta_j\} \tag{4.1}$$

b) for any j, the boundary values $b_{\operatorname{Im} z' \to 0}(h_j)$ and $b_{\operatorname{Im} z' \to 0}(h_{j+1})$ coincide as sections in $\mathcal{CO}(U' \times V_j'' \times \Gamma')$.

A regular sequence $\{h_j\}_j$ on $U' \times V'' \times \Gamma'$ defines a microfunction $v(x', z'')$ with holomorphic parameter z'' on $U' \times V'' \times \Gamma'$ in the following way. For any fixed point $\dot{q} = (\dot{x}', \dot{z}''; \dot{\xi}') \in U' \times V'' \times \Gamma'$, we can choose an index j with $\dot{z}'' \in V_j''$, and define a germ $v_{\dot{q}} = b_{\operatorname{Im} z' \to 0}(h_j)$ in $\mathcal{CO}_{\dot{q}}$. By the definition of regular sequences, this germ does not depend on the choice of j above, and the family $\{v_{\dot{q}}\}_{\dot{q} \in U' \times V'' \times \Gamma'}$ defines a global section v in $\mathcal{CO}(U' \times V'' \times \Gamma')$. We denote this v by $b_{\operatorname{Im} z' \to 0}(\{h_j\}_j)$.

In the case when we consider defining functions for second hyperfunctions, a typical and important case for regular sequences is the one when V'' and V_j'' have the form

$$V'' = \{z'' \in U'' + iG''; |\operatorname{Im} z''| < \delta\}, \tag{4.2}$$

$$V_j'' = \{z'' \in V''; |\operatorname{Im} z''| > 1/j\}, \tag{4.3}$$

with U'' some open set in $\mathbb{R}^{n-d}$, G'' an open convex cone in $\mathbb{R}^{n-d}$, and δ a positive constant. In this case it makes sense to consider the boundary value

$$u = b_{\operatorname{Im} z'' \to 0}(b_{\operatorname{Im} z' \to 0}(\{h_j\}_j)) \in \mathcal{B}^2_\Sigma(U' \times U'' \times \Gamma').$$

Note that $b_{\operatorname{Im} z' \to 0}(\{h_j\}_j)$ is a section in $\mathcal{CO}_N(U' \times V'' \times \Gamma')$. We denote this second hyperfunction u by $b^2(\{h_j\}_j)$.

Of course, in general, the fact that V'' has the form (4.2) does not by itself imply that V_j'' has the form (4.3). However, if we are only interested in the boundary value $b_{\operatorname{Im} z'' \to 0}(v) \in \mathcal{B}^2_\Sigma$ of v for x'' contained in a neighborhood of some fixed point $\dot{x}'' \in U''$, we may shrink U'', G'' and δ, and we may also renumber j. After having done this, we may then assume that V_j'' has the form (4.3). More precisely, we have

Lemma 4.2. *Let $\{h_j\}_{j \geq 1}$ be regular sequence on $U' \times V'' \times \Gamma'$ where the domain of holomorphy of each h_j is as in (4.1), definition 4.1. Assume that V'' has the form (4.2). Then for any positive $\tilde{\delta} < \delta$, any $\tilde{U}'' \subset\subset U''$ and any strict subcone $\tilde{G}''$ in G'', we can take a subsequence $\{h_{j_k}\}_{k=1,2,\ldots}$ which is a regular sequence on $U' \times \tilde{V}'' \times \Gamma'$ with*

$$\tilde{V}'' = \{z'' \in \tilde{U}'' + i\tilde{G}''; |\operatorname{Im} z''| < \tilde{\delta}\}$$

such that each h_{j_k} is holomorphic on

$$\{z = (z', z'') \in (U' + iG') \times \tilde{V}''; |\operatorname{Im} z'| < \delta_{j_k}, |\operatorname{Im} z''| > 1/k\},$$

and that the sequence $\{h_{j_k}\}_k$ defines the same microfunction with holomorphic parameter as $\{h_j\}_j$ on $U' \times \tilde{V}'' \times \Gamma'$.

Remark 4.3. *Since the microfunction $b_{\operatorname{Im} z'\to 0}(h_j)$ vanishes outside the set $\{\xi' \in G'^{\perp}\}$, there is no direct meaning in considering a regular sequence in the case when $\Gamma' \cap G'^{\perp} = \emptyset$. It is therefore worth noticing that in the definition, we have not asked for any relation between G' and Γ'. The reason for this is that we may want to restrict a sequence $\{h_j\}_j$ initially defined on $U' \times V'' \times \Gamma'$ (and for which we had that $\Gamma' \cap G'^{\perp} \neq \emptyset$) to the "same sequence", but defined on a smaller set of form $\tilde{U}' \times \tilde{V}'' \times \tilde{\Gamma}'$ with $\tilde{U}' \subset U'$, $\tilde{\Gamma}' \subset \Gamma'$ and $\tilde{V}'' \subset V''$.*

Trivial examples of regular sequences are constant sequences defined as follows.

Let $h(z)$ be a holomorphic function defined on a set of form

$$(U' + iG') \times V'' \cap \{|\operatorname{Im} z'| < \delta\}.$$

Also define $\{V_j''\}_j$, $\{\delta_j\}_j$, $\{h_j\}_j$ by

$$V_j'' = V'', \quad \delta_j = \delta, \quad h_j = h, \quad (j = 1, 2, \ldots).$$

Then the $\{h_j\}_j$ define a regular sequence of holomorphic functions. In this case, the microfunction v with holomorphic parameter associated with $\{h_j\}$ on $U' \times V'' \times \mathbb{R}^d$ is nothing but $b_{\operatorname{Im} z'\to 0}(h)$. Moreover if V'' has the form (4.2), then the boundary value $u = b_{\operatorname{Im} z''\to 0}(v) \in \mathcal{B}_{\Sigma}^2$ is classical.

Conversely, we can show, using remark 2.1, that any classical second hyperfunction u has, locally, a representation

$$u = \sum_k b_{\operatorname{Im} z''\to 0}(v_k)$$

with microfunctions with holomorphic parameter v_k's, where each v_k can be associated with a constant regular sequence.

For general second hyperfunctions, we give,

Theorem 4.4. *Let $u(x', x'') \in \mathcal{B}_{\Sigma,\dot{q}}^2$ be a second hyperfunction defined in a neighborhood of $\dot{q} = (\dot{x}', \dot{x}''; \dot{\xi}') \in \Sigma$. Then there exist regular sequences $\{h_j^k\}_j$ on a set of type*

$$U' \times (U'' + iG_k'') \times \Gamma' \cap \{|y''| < \delta\} \tag{4.4}$$

with $u = \sum_k b^2(\{h_j^k\}_j)$, where $U' \times U'' \times \Gamma'$ is a neighborhood of $\dot{q}$ and G_k'''s are open convex cones in $\mathbb{R}^{n-d}$.

Remark 4.5. *Let $K'' \subset \mathbb{R}^{n-d}$ be a compact set and consider a second hyperfunction u defined in a neighborhood of $\{(\dot{x}', x''; \dot{\xi}'); x'' \in \mathbb{R}^{n-d}\}$ with some fixed $\dot{x}' \in \mathbb{R}^d$ and $\dot{\xi}' \in \dot{\mathbb{R}}^d$ satisfying* $\operatorname{supp} u \subset \{x'' \in K''\}$. *In this case we can take the regular sequences $\{h_j^k\}_j$ in theorem 4.4 on some larger domains than those considered in (4.4). For example, if we fix subsets $\tilde{K}''$ and U'' in $\mathbb{R}^{n-d}$ with $\tilde{K}'' \subset\subset U'' \subset\subset \mathbb{R}^{n-d}$, we can take regular sequences $\{h_j^k\}_j$ on $U' \times V''^k \times \Gamma'$, where each $V''^k \subset \mathbb{C}^{n-d}$ includes not only $(U'' + iG_k'') \cap \{|y''| < \delta\}$ but also $U'' \setminus \tilde{K}''$. This fact can be proved with the aid of proposition 3.1.*

5 Weight functions

In the following we shall consider weighted L^∞-spaces. (For the terminology also cf. remark 5.3.) The weights will be of two types, associated roughly speaking with first and second microlocalization. These weights will be defined on sets of form:

$$(\Gamma' \times \mathbb{R}^{n-d}) \cap \{\xi; |\xi''| < \delta|\xi'|\}, \text{ or } (\Gamma' \times \Gamma'') \cap \{\xi; |\xi''| < \delta|\xi'|\}, \tag{5.1}$$

where $\Gamma' \subset \mathbb{R}^d$, $\Gamma'' \subset \mathbb{R}^{n-d}$, are open convex cones in $\mathbb{R}^d$, respectively $\mathbb{R}^{n-d}$ and δ is strictly positive (or sometimes, $\delta = +\infty$). In some cases we shall also allow for $\Gamma' = \mathbb{R}^d$ or $\Gamma' = \dot{\mathbb{R}}^d$. We shall denote by

$$\mathcal{H} = \{\ell : \Gamma' \times \mathbb{R}^{n-d} \to \mathbb{R}_+; \forall \varepsilon, \exists C_\varepsilon \ s.t.\ \ell(\xi) \le \varepsilon|\xi| + C_\varepsilon, \text{ if } \xi \in \Gamma' \times \mathbb{R}^{n-d}\}, \tag{5.2}$$

and by

$$\mathcal{F} = \{\varphi : \Gamma' \times \mathbb{R}^{n-d} \to \mathbb{R}_+; \forall j \in \mathbb{N}, \exists \delta_j > 0, \exists C_j > 0 \ s.t.\ \varphi(\xi) \le |\xi''|/j,$$
$$\text{if } |\xi''| \le \delta_j|\xi| \text{ and } |\xi| \ge C_j\}.$$

Functions in $\mathcal{F}$ shall be referred to as "sublinear". When we use sublinear functions to describe some geometry, the situation will become somewhat simpler if we assume that they depend only on the variable ξ'. It is then useful to note that for $\ell \in \mathcal{H}$ and $\delta > 0$, the function $\ell'(\xi') = \sup_{|\xi''| \le \delta|\xi'|} \ell(\xi', \xi'')$ is sublinear and satisfies $\ell(\xi) \le \ell'(\xi')$ on $\{\xi; \xi' \in \Gamma', |\xi''| \le \delta|\xi'|\}$. As a consequence we may assume in the sequel always that ℓ depends only on ξ'.

It is also clear that it is no loss of generality to assume that the sequences $j \to \delta_j$, which appear in the definition of $\mathcal{F}$ are strictly decreasing to zero and we shall do so in the following without further mention.

Lemma 5.1. *a) Let $\delta_j \searrow 0$ and $C_j \nearrow \infty$. Then there is a sublinear function ρ such that*

$$\{\xi \in \mathbb{R}^n; |\xi'| \le C_j, \delta_{j+1}|\xi'| \le |\xi''| \le \delta_j|\xi'|\} \subset \{\xi \in \mathbb{R}^n; |\xi''| \le \rho(\xi')\}. \tag{5.3}$$

b) Conversely, if we are given a sublinear function ρ and a sequence $\delta_j \searrow 0$, then there is a sequence $C_j \nearrow \infty$ such that

$$\{\xi \in \mathbb{R}^n; |\xi''| \le \rho(\xi'), |\xi''| < \delta_1|\xi'|\} \subset \bigcup_{j=1}^{\infty} \{\xi \in \mathbb{R}^n; |\xi'| \le C_j, \delta_{j+1}|\xi'| \le |\xi''| \le \delta_j|\xi'|\}.$$

Proof. a) It is no loss of generality to assume that $\delta_j C_j \nearrow \infty$. We define ρ by the conditions:

$$\rho(\xi) = C_j\delta_j \text{ if } |\xi'| \ge C_{j-1}, \delta_{j+1}|\xi'| \le |\xi''| < \delta_j|\xi'| \text{ or if } C_{j-1} \le |\xi'| \le C_j,\ |\xi''| \le \delta_{j+1}|\xi'|.$$

To prove (5.3) assume then by contradiction that $|\xi''| > \rho(\xi')$ and that $C_{s-1} \leq |\xi'| \leq C_s$, $\delta_{j+1}|\xi'| \leq |\xi''| \leq \delta_j|\xi'|$, for some j and s, $s \leq j$. Then $\rho(\xi') = C_s\delta_s$, so $\delta_j|\xi'| \geq |\xi''| > \rho(\xi') = C_s\delta_s$. This contradicts $|\xi'| \leq C_s$ since $\delta_s/\delta_j \leq 1$. This gives (5.3). Let us also check that ρ is sublinear. Let us then fix ε and assume that k is chosen with $\delta_k \leq \varepsilon$. It is clear that $\rho(\xi') \leq \varepsilon|\xi'|$ for $|\xi''| \leq \delta_k|\xi'|$. On the other hand on $|\xi''| \geq \delta_k|\xi'|$, ρ is bounded by $\max_{j\leq k} C_j\delta_j$, so $\rho(\xi') \leq \varepsilon|\xi'| + \max_{j\leq k}\delta_j C_j$.

b) It suffices to choose C_j so that $\rho(\xi') \leq \delta_{j+1}|\xi'|/2$ for $|\xi'| \geq C_j$. Indeed, in that case $|\xi''| \leq \rho(\xi')$ is not compatible with $\delta_{j+1}|\xi'| \leq |\xi''|$ if $|\xi'| \geq C_j$. □

2. We next consider the following definition

Definition 5.2. *a) Consider some open convex cone $\Gamma' \subset \mathbb{R}^d$. We denote by $\mathcal{M}^2(\Gamma')$ (the "two" comes from "second") the space of measurable functions $\mu : \Gamma' \times \mathbb{R}^{n-d} \to \mathbb{C}$ such that we can find sublinear functions ℓ, ρ and $\varphi \in \mathcal{F}$ so that*

$$|\mu(\xi)| \leq \exp[\ell(\xi') + \varphi(\xi)], \ \text{if } \xi' \in \Gamma', |\xi''| \geq \rho(\xi'), |\xi''| < \delta|\xi'|. \tag{5.4}$$

b) Two functions $\mu \in \mathcal{M}^2(\Gamma')$ and $\tilde{\mu} \in \mathcal{M}^2(\tilde{\Gamma}')$ will be called equivalent on $\Gamma' \cap \tilde{\Gamma}'$ if we can find $c > 0$, sublinear functions ℓ', ρ', and $d > 0$ so that

$$|\mu(\xi) - \tilde{\mu}(\xi)| \leq \exp[\ell'(\xi') - d|\xi''|], \ \forall \xi' \in \Gamma' \cap \tilde{\Gamma}', |\xi''| \geq \rho'(\xi'), |\xi''| < c|\xi'|. \tag{5.5}$$

Remark 5.3. *a) In some arguments, it seems more appropriate to replace pointwise inequalities as in (5.4) and (5.5) by similar L^2-type inequalities. This is of course also the main reason of why we call the functions in $\mathcal{H}$ and $\mathcal{F}$ "weights". However, in the present paper we have no particular advantage from working with L^2-inequalities, and will not do it.*

b) There is no deep meaning in the presence of the term $l'(\xi)$ in the exponential in (5.5). Indeed, we can alternatively ask that for some suitable sublinear function ρ we have

$$|\mu(\xi) - \tilde{\mu}(\xi)| \leq \exp[-d|\xi''|], \ \text{if } \xi \in \Gamma' \cap \tilde{\Gamma}', |\xi''| > \rho(\xi'), |\xi''| < c|\xi'|.$$

In a future paper it will turn out that $\mathcal{M}^2(\Gamma')$ corresponds roughly speaking to the space of Fourier-transformed of second hyperfunctions. We shall also need a corresponding space in the case of second microfunctions.

Definition 5.4. *Consider some open convex cone $\Gamma' \subset \mathbb{R}^d$ and some open cone $\Gamma'' \subset \mathbb{R}^{n-d}$. We denote by $\mathcal{M}^2(\Gamma', \Gamma'')$ the space of measurable functions $\mu : \Gamma' \times \Gamma'' \to \mathbb{C}$ such that we can find sublinear functions ℓ, ρ and $\varphi \in \mathcal{F}$ so that (5.4) holds, if we add to it the condition $\xi'' \in \Gamma''$. Moreover, two functions $\mu \in \mathcal{M}^2(\Gamma', \Gamma'')$ and $\tilde{\mu} \in \mathcal{M}^2(\tilde{\Gamma}', \tilde{\Gamma}'')$ will be called equivalent on $(\Gamma' \cap \tilde{\Gamma}') \times (\Gamma'' \cap \tilde{\Gamma}'')$ if (5.5) holds if we add the condition $\xi'' \in \Gamma'' \cap \tilde{\Gamma}''$ there.*

To state our next result we introduce a notation. To do so, we shall fix a sequence, $\delta_j \searrow 0$ and denote by

$$S_j = \{\xi; \delta_{j+1}|\xi'| \leq |\xi''| \leq \delta_j|\xi'|\}. \tag{5.6}$$

We shall have sometimes to assume that $\delta_j \to 0$ sufficiently rapidly, so the δ_j may vary from argument to argument and the notation S_j will always refer to the sequence δ_j just considered.

Remark 5.5. *In view of lemma 5.1 the following two statements are equivalent if we are given some sequence $\delta_j \searrow 0$:*

a) there is a sublinear function ρ so that some given property $\mathcal{P}$ holds for $|\xi''| \geq \rho(\xi')$,

b) there is a sequence $C_j \to \infty$ such that $\mathcal{P}$ holds for $\xi \in S_j$ if $|\xi'| \geq C_j$, whatever j is.

We have the following estimate:

Proposition 5.6. *Consider $\mu \in \mathcal{M}^2(\Gamma')$. If $C_j \to \infty$ sufficiently rapidly, then we will have*

$$|\mu(\xi)| \leq \exp[2|\xi''|/j], \text{ for } \xi \in S_j, |\xi| > C_j, |\xi''| > \rho(\xi). \tag{5.7}$$

Proof. Let $\varphi \in \mathcal{F}$ and a sublinear ℓ be given such that $|\mu(\xi)| \leq \exp[\ell(\xi') + \varphi(\xi)]$ for $\xi' \in \Gamma'$, $|\xi''| > \rho(\xi')$, $|\xi''| \leq \delta|\xi'|$. In view of our notations, $\varphi(\xi) \leq |\xi''|/j$ if $\xi \in S_j$ and $|\xi| \geq C_j$, so that $\mu(\xi) \leq \exp[\ell(\xi') + |\xi''|/j]$ when $\xi \in S_j$, $|\xi| \geq C_j$. The proposition will follow if we can show that $\ell(\xi) \leq |\xi''|/j$ for $\xi \in S_j$ and $|\xi| \geq C_j$, provided C_j is sufficiently large. This is an immediate consequence of the sublinearity of ℓ when combined with the fact that on S_j $|\xi''| \geq \delta_{j+1}|\xi'|$. □

Lemma 5.7. *Let $\ell_j, j \geq 1$, be a sequence of sublinear functions. Then there is a sublinear function ℓ and a sequence of constants C_j so that $\ell_j(\xi') \leq \ell(\xi')$ for $|\xi'| \geq C_j$, $\forall j$.*

Proof. We choose constants C_{jk} so that

$$\ell_j(\xi') \leq |\xi'|/k \text{ if } |\xi'| \geq C_{jk}.$$

We also define C_k by $C_k = \max_{j \leq k} C_{jk}$ and set $\ell(\xi') = |\xi'|/k$ if $C_k \leq |\xi'| < C_{k+1}$. It is then clear that ℓ is sublinear and it remains to check that $\ell_j(\xi') \leq \ell(\xi')$ if $|\xi'| > C_j$. In fact, for $C_s \leq |\xi'| < C_{s+1}$, $s \geq j$, we will have that $\ell(\xi') = |\xi'|/s$. On the other hand, $C_s \geq C_{js}$, so for $|\xi'| \geq C_s$ we have $\ell_j(\xi') \leq |\xi'|/s$. This concludes the proof. □

Lemma 5.8. *Let $\varphi \in \mathcal{F}$ and fix ρ sublinear. Then φ is sublinear on the set*

$$A = \{\xi \in \Gamma; |\xi''| \leq \rho(\xi')\}. \tag{5.8}$$

Proof. By the definition of $\mathcal{F}$, there are δ_j, C_j, so that $\varphi(\xi) \le |\xi''|/j + C_j$ if $|\xi''| \le \delta_j|\xi'|$. However, on A, $|\xi''| \le \delta_j|\xi'|$ if $|\xi'|$ is sufficiently large, so we obtain $\varphi(\xi) \le |\xi''|/j + C_j$, if $|\xi'|$ is sufficiently large. This gives the desired statement. □

Proposition 5.9. *Let $\delta_j \searrow 0$, $\Gamma'_+ \subset\subset \Gamma'$, $\Gamma''_+ \subset\subset \Gamma''$, and ρ be given. Then there exist C^∞- functions e_j and a sublinear function ρ_+ such that:*

a) $e_j(\xi) = 1$ if $\xi' \in \Gamma'_+$, $\xi'' \in \Gamma''_+$, $|\xi''| \le \delta_j|\xi'|/2$, $|\xi''| \ge \rho_+(\xi')$,

b) $e_j(\xi) = 0$, when $|\xi''| \le \rho(\xi')$, or $\xi' \notin \Gamma'$, or $\xi'' \notin \Gamma''$ or $|\xi''| \ge \delta_j|\xi'|$,

c) $|\nabla e_j(\xi)| \le c$.

In particular, the e_j vanish of infinite order on the boundary of $\{\xi; \xi' \in \Gamma', \xi'' \in \Gamma'', |\xi''| \le \delta_j|\xi'|, |\xi''| \ge \rho(\xi')\}$.

Proof. If the conclusion of the proposition holds for some ρ_+, it holds for any larger ρ_+. We may therefore choose ρ_+ in various "subcases" during the argument independently. We next denote by A_j, respectively A_j^+, the sets:

$$A_j = \{\xi; \xi' \in \Gamma', \xi'' \in \Gamma'', |\xi''| \le \delta_j|\xi'|, |\xi''| \ge \rho(\xi')\},$$

$$A_j^+ = \{\xi; \xi' \in \Gamma'_+, \xi'' \in \Gamma''_+, |\xi''| \le \delta_j|\xi'|/2, |\xi''| \ge \rho_+(\xi')\}.$$

It suffices to show that if ρ_+ is chosen suitably, then dist $(A_j^+, \complement A_j) \ge c'$ for some c' which does not depend on j. In view of lemma 5.1 this will be the case, if we can find a sequence $C'_j \nearrow \infty$ such that if we denote $A_j'^+$ the set

$$\{\xi \in \Gamma'_+ \times \Gamma''_+; |\xi''| \le \delta_j|\xi'|/2, |\xi'| \ge C'_s \text{ provided } \delta_{s+1}|\xi'| \le |\xi''| \le \delta_s|\xi'| \text{ for some } s\},$$

then dist $(A_j, \complement A_j'^+) \ge c''$. To see that such a sequence C'_j exists, we fix a constant c_1 so that $|\xi' - \eta'| \le c_1|\xi'|$, respectively $|\xi'' - \eta''| \le c_1|\xi''|$, implies $|\xi'| \le (1 + 1/4)|\eta'|$, $|\eta'| \le (1 + 1/4)|\xi'|$, $|\xi''| \le (1 + 1/4)|\eta''|$, $|\eta''| \le (1 + 1/4)|\xi''|$. It folows in particular that if simultaneouly $|\xi' - \eta'| \le c_1|\xi'|$ and $|\xi'' - \eta''| \le c_1|\xi''|$, then $|\eta''| \le \delta_j|\eta'|/2$ implies $|\xi''| \le \delta_j|\xi'|$. Also fix a constant c_2 such that $\eta' \in \Gamma'_+$, $\xi' \notin \Gamma'$, repsectively $\eta'' \in \Gamma''_+$, $\xi'' \notin \Gamma''$, implies $|\xi' - \eta'| \ge c_2(|\xi'| + |\eta'|)$, $|\xi'' - \eta''| \ge c_2(|\xi''| + |\eta''|)$. Assume now that $\eta \in A_j'^+$ and that $\xi \notin A_j$. We want to show that then $|\xi - \eta| \ge c$ for some c which does not depend on j. The condition $\xi \notin A_j$ means that at least one of the following holds: i) $\xi' \notin \Gamma'$, ii) $\xi'' \notin \Gamma''$, iii) $|\xi''| \ge \delta_j|\xi'|$, or iv) $|\xi''| \le \rho(\xi')$. In the case i), we thus have $\eta' \in \Gamma'_+, \xi' \notin \Gamma'$. We therefore have $|\xi' - \eta'| \ge c_2|\xi'|$. This gives $|\xi - \eta| \ge c$, if we assume that $|\eta'| \ge C'_j$ and C'_j is large enough. A similar argument also holds in case ii). Assume now $|\xi''| \le \delta_j|\xi'|$. In this case however, we have seen that we must have one of the following two inequalities: $|\xi' - \eta'| \ge c_1|\xi'|$ or $|\xi'' - \eta''| \ge c_1|\xi''|$. Since we can now continue as above, this shows that also case c) can be treated, increasing, if need be, the C'_j. We are are then left with the case $|\xi''| \leq \rho(\xi')$. Also in this case only $|\xi' - \eta'| \leq c_1|\xi'|$ and $|\xi'' - \eta''| \leq c_1|\xi''|$ may cause problems. However, if we choose ρ_+ such that $\rho_+(\xi') \geq 2\sup_{|\eta'-\xi'|\leq c_1|\xi'|}\rho(\eta')$, then we cannot have simultaneously $|\eta''| \geq \rho_+(\eta')$, $|\xi''| \leq \rho(\xi')$, $|\xi' - \eta'| \leq c_1|\xi'|$ and $|\xi'' - \eta''| \leq c_1|\xi''|$. This conludes the argument. □

6 Regular sequences of measurable functions

In addition to regular sequences of holomorphic functions we shall also consider regular sequences of measurable functions. These are sequences of measurable functions $\mu_j : \Gamma' \times \Gamma'' \to \mathbb{C}$ such that we can find $\delta_j \searrow 0$, $d > 0$, and sublinear functions ℓ_j, ρ, such that

$$|\mu_j(\xi)| \leq \exp[\ell_j(\xi') + 2|\xi''|/j], \xi' \in \Gamma', \xi'' \in \Gamma'', |\xi''| \geq \rho(\xi'), |\xi''| \leq \delta|\xi'|, \tag{6.1}$$

$$|\mu_j(\xi) - \mu_k(\xi)| \leq \exp[\ell_j(\xi') - d|\xi''|], \text{ if } \xi' \in \Gamma', \xi'' \in \Gamma'', \rho(\xi') < |\xi''| < \delta_j|\xi'|, k \leq j. \tag{6.2}$$

We shall say that two such sequences $\{\mu_j\}_{j\geq 1}$ and $\{\tilde{\mu}_j\}_{j\geq 1}$ are equivalent and write

$$\{\mu_j\}_{j\geq 1} \sim \{\tilde{\mu}_j\}_{j\geq 1}$$

if we can find $\delta_j \searrow 0$, ρ sublinear, $d > 0$, so that

$$|\mu_j(\xi) - \tilde{\mu}_j(\xi)| \leq \exp[\ell_j(\xi) - d|\xi''|], \text{ if } \xi' \in \Gamma', \xi'' \in \Gamma'', \rho(\xi') \leq |\xi''| < \delta_j|\xi'|. \tag{6.3}$$

Here Γ' and Γ'' are open cones in $\mathbb{R}^d$, respectively $\mathbb{R}^{n-d}$. The cone Γ' will usually be convex, but the cone Γ'' could in principle be $\dot{\mathbb{R}}^{n-d}$ or $\mathbb{R}^{n-d}$ itself. (This case will correspond to second hyperfunctions, whereas the case Γ'' convex corresponds roughly speaking to second microfunctions.) In the remainder of this section we shall always work with $\Gamma'' = \mathbb{R}^{n-d}$. The case of a smaller Γ'' gives no additional difficulties.

Remark 6.1. *The condition (6.1) can be replaced by the following: there are $\delta_j \searrow 0, \ell_j, \rho$ and a constant $\delta > 0$ such that*

$$|\mu_j(\xi)| \leq \exp[\ell_j(\xi') + 2|\xi''|/j], \xi' \in \Gamma', \xi'' \in \Gamma'', \rho(\xi') \leq |\xi''| \leq \delta|\xi'|. \tag{6.4}$$

It is then clear that all relevant conditions are given on sets of form $\{\xi; \xi' \in \Gamma', \xi'' \in \Gamma'', \rho(\xi') \leq |\xi''| \leq \delta|\xi'|\}$ and in particular, there is no reason for the μ_j to be defined outside such sets. However, also the "converse" will sometimes be useful: the μ_j are defined on all of $\Gamma' \times \Gamma''$ and satisfy the condition:

$$|\mu_j(\xi)| \leq \exp[\ell_j(\xi') + 2|\xi''|/j], \text{ if } \xi' \in \Gamma', \xi'' \in \Gamma'', j \geq 1, \tag{6.5}$$

rather than (6.1).

Trivial examples of regular sequences of measurable functions are the "constant" sequences obtained in the following way: we start from a given $\mu \in \mathcal{M}^2(\Gamma')$ and consider $\delta \searrow 0$, $C_j \nearrow \infty$ sufficiently rapidly. We then set $\mu_j(\xi) = 0$ if $|\xi''| > \delta_j|\xi'|$, or $|\xi| \leq C_j$, $\mu_j(\xi) = \mu(\xi)$ if $|\xi''| \leq \delta_j|\xi'|$ and $|\xi| > C_j$. It is immediate that this defines a regular sequence. Indeed, for a suitable choice of δ_j and C_j we shall have $|\mu_j(\xi)| \leq \exp[\ell(\xi') + |\xi''|/j]$ for large ξ and the fact that $\mu_j(\xi) - \mu_k(\xi)$ vanishes for $|\xi''| \leq \delta_j|\xi'|$ when $k < j$ leads to (6.2). Moreover, if μ_j and $\tilde{\mu}_j$ are two regular sequences associated with some given μ in this way, then $\{\mu_j\}_{j\geq 1} \sim \{\tilde{\mu}_j\}_{j\geq 1}$, as is easy to see. If we identify $\mu \in \mathcal{M}^2(\Gamma')$ with the regular sequence $\{\mu_j\}_{j\geq 1}$ constructed just before, it makes then sense to consider the relations $\mu \sim \tilde{\mu}$ and $\mu \sim \{\nu_j\}_{j\geq 1}$ for $\mu, \tilde{\mu} \in \mathcal{M}^2(\Gamma')$ and a regular sequence of measurable functions $\{\nu_j\}_{j\geq 1}$. It is immediate that the equivalence relation $\mu \sim \tilde{\mu}$ defined in this way coincides with the one introduced in section 5. We have in fact the following simple:

Lemma 6.2. *a) Let μ be given in $\mathcal{M}^2(\Gamma')$ and assume that $\mu \sim \{\nu_j\}_{j\geq 1}$ for some regular sequence ν_j. Then we can find $d > 0$, a sequence $\delta_j \searrow 0$ and sublinear functions ρ, ℓ_j such that*

$$|\mu(\xi) - \nu_j(\xi)| \leq \exp[\ell_j(\xi') - d|\xi''|], \ \text{if } \rho(\xi') \leq |\xi''| \leq \delta_j|\xi'|.$$

b) Let $\mu, \tilde{\mu}$ be given in $\mathcal{M}^2(\Gamma')$. Then $\mu \sim \tilde{\mu}$ in the sense just introduced, if and only if we can find $c > 0$, sublinear functions ρ, ℓ, and $d > 0$ so that

$$|\mu(\xi) - \tilde{\mu}(\xi)| \leq \exp[\ell(\xi') - d|\xi''|], \forall \xi \in \Gamma', \rho(\xi') \leq |\xi''| < \delta|\xi'|. \qquad (6.6)$$

Proof. a) This is immediate from the definition.

b) Let us consider some sequences $\delta_j \searrow 0$, $C_j \nearrow \infty$ (sufficiently rapidly) and denote by $\{\mu_j\}_{j\geq 1}$, $\{\tilde{\mu}_j\}_{j\geq 1}$ the constant regular sequences associated with μ and $\tilde{\mu}$ as before. The assumption is that $\{\mu_j\}_{j\geq 1} \sim \{\tilde{\mu}_j\}_{j\geq 1}$. In particular we have that on $\xi' \in \Gamma', \xi \in S_j$, $|\mu(\xi) - \tilde{\mu}(\xi)| = |\mu_j(\xi) - \tilde{\mu}_j(\xi)| \leq \exp[\ell_j(\xi') - d|\xi''|] \leq \exp[\ell(\xi') - d|\xi''|]$ if ξ is large. This gives (6.6) with $\delta = \delta_1$ □

We can also associate elements in $\mathcal{M}^2(\Gamma')$ with regular sequences:

Theorem 6.3. *Let $\{\mu_j\}_j$ be a regular sequence of measurable functions on $\Gamma' \times \mathbb{R}^{n-d}$. For each fixed j^0 the function $\mu = \mu_{j^0}$ belongs then to $\mathcal{M}^2(\Gamma')$ and satisfies that $\mu \sim \{\mu_j\}_{j\geq 1}$*

Proof. We fix j_0 and denote $\mu = \mu_{j_0}$. Once we prove that $\mu \in \mathcal{M}^2(\Gamma')$, it is immediate that $\mu \sim \{\mu_j\}_{j\geq 1}$. It will thus suffice to prove that $\mu \in \mathcal{M}^2(\Gamma')$.

We use here that $\{\mu_j\}_j$ forms a regular sequence of measurable functions. We may therefore assume that there exist $\delta_j \searrow 0$, $d > 0$, and sublinear functions ℓ_j, ρ satisfying the condition (6.5) and the condition

$$|\mu_j(\xi) - \mu_k(\xi)| \leq \exp\left[\ell_j(\xi') - d|\xi''|\right], \text{ if } \xi' \in \Gamma', \rho(\xi') < |\xi''| < \delta_j|\xi'|, k \leq j.$$

By replacing $\ell_j(\xi')$ by the function $\max_{1\leq k\leq j} \ell_j(\xi')$, we may assume, from the beginning, that $\ell_j(\xi') \leq \ell_{j+1}(\xi')$ for any j and ξ'. Also recall that what we want to do is to find a sublinear function ℓ and $\varphi \in \mathcal{F}$ for which the condition (5.4) in definition 5.2 holds.

To find ℓ, we observe that for a sequence of sublinear functions $\{\ell_j\}_j$, we can find, (using lemma 5.7) a sublinear function ℓ and a sequence of positive numbers $\{C_j\}_j$ such that $\ell_j(\xi') \leq \ell(\xi')$ holds for any j and any ξ' with $|\xi'| \geq C_j$.

Next we define $\varphi(\xi)$ by

$$\varphi(\xi) = \begin{cases} \log|\mu(\xi)| - \ell(\xi'), & \rho(\xi') < |\xi''| \\ 0, & \rho(\xi') \geq |\xi''|. \end{cases}$$

The proof will come to an end if we can show that $\varphi \in \mathcal{F}$. For any $j > j_0$ and ξ with $\rho(\xi') < |\xi''| < \delta_j|\xi'|$ and $|\xi'| \geq C_j$, we have in fact the following two estimates

$$|\mu(\xi) - \mu_j(\xi)| \leq \exp\left[\ell_{j_0}(\xi') - d|\xi''|\right],$$
$$|\mu_j(\xi)| \leq \exp\left[\ell_j(\xi') + 2|\xi''|/j\right],$$

which imply

$$|\mu(\xi)| \leq 2\exp\left[\ell(\xi') + 2|\xi''|/j\right].$$

Thus we have

$$\varphi(\xi) \leq 2|\xi''|/j + \log 2 \text{ if } \rho(\xi') < |\xi''| < \delta_j|\xi'|, |\xi'| \geq C_j,$$

which already shows that $\varphi \in \mathcal{F}$ (if we replace the sequences $\{\delta_j\}_j$ and $\{C_j\}_j$ by their subsequences $\{\delta_{2j}\}_j$ and $\{C_{2j}\}_j$ respectively). □

7 The Fourier-inverse transform

Let $\Gamma' \subset \mathbb{R}^d$, $\Gamma'' \subset \mathbb{R}^{n-d}$ be open convex cones, consider a sublinear function $\rho : \mathbb{R}^d \to \mathbb{R}$ and let $\mu : \{\xi \in \mathbb{R}^n; \xi' \in \Gamma', \xi'' \in \Gamma'', \rho(\xi') \leq |\xi''| \leq \delta|\xi'|\} \to \mathbb{C}$ be a measurable function. We shall assume for simplicity, if not specified otherwise, that Γ' and Γ'' are proper cones. We shall denote the set $\{\xi \in \mathbb{R}^n; \xi' \in \Gamma', \xi'' \in \Gamma'', \rho(\xi') \leq |\xi''| \leq \delta|\xi'|\}$ in the sequel by A and assume that for some sublinear function ℓ and some $\varphi \in \mathcal{F}$

$$|\mu(\xi)| \leq \exp\left[\ell(\xi') + \varphi(\xi'')\right], \xi \in A, \tag{7.1}$$

i.e., $\mu \in \mathcal{M}^2(\Gamma', \Gamma'')$. We can then choose $\delta_j \searrow 0$ and $C_j \nearrow \infty$ so that

$$|\mu(\xi)| \leq \exp[\ell(\xi') + |\xi''|/j] \text{ if } \xi \in A, |\xi''| \leq \delta_j|\xi'|, |\xi| \geq C_j. \tag{7.2}$$

We want to give a meaning to the integral

$$\int \exp[i\langle x, \xi\rangle]\mu(\xi)\, d\xi, \tag{7.3}$$

which, apart from a multiplicative factor $(2\pi)^{-n}$, should be the Fourier-inverse of μ. Since the integrand $\mu(\xi)\exp[i\langle x, \xi\rangle]$ is not in $L^1(\mathbb{R}^n)$ for fixed $x \in \mathbb{R}^n$, we have to indicate a regularization procedure for (7.3). Actually, the regularization of (7.3) will be in second microfunctions and no meaning will be given to the integral in (7.3) for fixed real x. We consider the functions h_j defined formally by

$$h_j(z) = \int_{\xi \in A, |\xi''| \leq \delta_j|\xi'|} \exp[i\langle z, \xi\rangle]\mu(\xi)\, d\xi. \tag{7.4}$$

Under suitable additional assumptions h_j will be holomorphic on $\mathbb{R}^n + iG_j$ with

$$G_j = \operatorname{Int}\big(\Gamma' \times \Gamma'' \cap \{|\xi''| \leq \delta_j|\xi'|\}\big)^{\perp},$$

this is e.g. the case when $|\mu(\xi)| \leq \exp[\ell(\xi')]$ for $\xi \in A$. Regularization of $\mathcal{F}^{-1}(\mu)$ will then essentially be in classical second hyperfunctions. We shall briefly discuss this situation later on and it has also been discussed in [12]. However, in the general case, the domain of holomorphy of h_j will be smaller and in particular remain at some distance from the real space, due to the factor $\exp(|\xi''|/j)$ in (7.2) which is of exponential growth type.

To calculate reasonably large domains of holomorphy, let us fix a vector $\dot{y}'' \in \operatorname{Int}\Gamma''^{\perp}$ with $\inf_{\xi'' \in \Gamma''}\langle \dot{y}'', \xi''/|\xi''|\rangle = 1$ and consider the sets

$$V_j = \mathbb{R}^n + i((0, \dot{y}''/j) + G_j), \quad j = 1, 2, \ldots.$$

The integral in (7.4) is then defined for $z \in V_j$ and defines an analytic function there. In fact, we have, for any $y = (0, \dot{y}''/j) + \tilde{y}$ with $\tilde{y} \in G_j$, the estimate

$$\langle y, \xi\rangle \geq |\xi''|/j + \langle \tilde{y}, \xi\rangle, \text{ if } \xi \in \Gamma' \times \Gamma'', |\xi''| \leq \delta_j|\xi'|.$$

Therefore, if we fix some compact set $K \subset G_j$, then we obtain at first that $\langle \tilde{y}, \xi\rangle \geq c|\xi|$, $\forall \tilde{y} \in K$, $\forall \xi \in \Gamma' \times \Gamma''$, $|\xi''| \leq \delta_j|\xi'|$, for some $c > 0$ which depends only on K and then that

$$|\exp[i\langle z, \xi\rangle]\mu(\xi)| \leq \exp[\ell(\xi') - \langle \tilde{y}, \xi\rangle] \leq \exp[\ell(\xi') - c|\xi|], \quad \tilde{y} = \operatorname{Im} z - (0, \dot{y}''/j) \in K,$$

if $\xi \in A$, $|\xi''| \leq \delta_j|\xi'|$, $|\xi| \geq C_j$. This shows that the integrand in (7.4) has an exponential decay estimate uniformly in z on any compact set in V_j.

Now define $G' = \operatorname{Int} \Gamma'^{\perp}$, $G'' = \operatorname{Int} \Gamma''^{\perp}$. Then G_j includes $G' \times G''$ and G_j increases with j. Moreover if we set $V'' = \mathbb{R}^{n-d} + iG''$ and $V_j'' = \mathbb{R}^{n-d} + i(\dot{y}''/j + G'')$, we have that V_j'' also increases and exhausts V'', i.e., $V'' = \bigcup_j V_j''$. It follows in particular from our discussion that the h_j are holomorphic on $(\mathbb{R}^d + iG') \times V_j''$.

Lemma 7.1. $\{h_j\}_j$ *forms a regular sequence on* $(\mathbb{R}^d + i\Gamma') \times V''$.

Proof. We have already observed that $V'' = \bigcup_j V_j''$ and can argue as above to show that the difference

$$h_j(z) - h_{j+1}(z) = \int_{\xi \in A, \delta_{j+1}|\xi'| \leq |\xi''| \leq \delta_j |\xi'|} \exp[i\langle z, \xi\rangle] \mu(\xi)\, d\xi$$

is analytic on the set $\mathbb{R}^n + i((0, \dot{y}''/j) + G_{j,j+1})$, with

$$G_{j,j+1} = \operatorname{Int}(\{\xi \in \Gamma' \times \Gamma''; \delta_{j+1}|\xi'| \leq |\xi''| \leq \delta_j|\xi'|\}^{\perp}).$$

Thus it suffices to show that $G_{j,j+1} \supset \{0\} \times G''$. For any $y'' \in G''$, there exists a positive constant c with

$$\langle y'', \xi''/|\xi''|\rangle \geq c, \text{ if } \xi'' \in \Gamma''.$$

Then for any $\xi \in \Gamma' \times \Gamma''$ with $\delta_{j+1}|\xi'| \leq |\xi''|$,

$$\langle (0, y''), \xi/|\xi|\rangle \geq c|\xi''|/|\xi| \geq c\delta_{j+1}/(\delta_{j+1} + 1)$$

holds and implies that $(0, y'')$ belongs to $G_{j,j+1}$. □

Denote by $u = (2\pi)^{-n} \operatorname{sp}^2_{\Sigma}(b^2(\{h_j\}_{j\geq 1}))$. It is immediate that the second micro-function u does not depend on the choice of the δ_j. This shows that u is associated directly with μ. We shall call u the Fourier-inverse of μ and write

$$u = \mathcal{F}^{-1}(\mu),$$

or sometimes

$$u = \mathcal{F}^{-1}_{\Gamma' \times \Gamma''}(\mu),$$

if we want to make Γ' and Γ'' explicit in the notation. This $\mathcal{F}^{-1}(\mu)$ is defined on $\mathbb{R}^d \times \dot{\mathbb{R}}^d \times \mathbb{R}^{n-d} \times \dot{\mathbb{R}}^{n-d}$, and satisfies

$$\operatorname{supp} \mathcal{F}^{-1}(\mu) \subset \mathbb{R}^d \times \overline{\Gamma'} \times \mathbb{R}^{n-d} \times \overline{\Gamma''}.$$

Note that the second hyperfunction $b^2(\{h_j\}_{j\geq 1})$ does not depend on the choice of δ_j. However, if in the definition of the set A, we replace ρ by some other sublinear function ρ', ρ' larger than ρ, then the regular sequence defined by (7.4) will change, and the difference gives a non-zero contribution as a second hyperfunction. (Actually this difference belongs, in general, to $\mathcal{A}^2_\Sigma$). This is the reason why we have defined the Fourier-inverse transform $\mathcal{F}^{-1}(\mu)$ as a second microfunction and not as a second hyperfunction. Also see remark 7.3 later on.

We shall also consider the corresponding situation when we work with $\mu : \{\xi \in \mathbb{R}^n; \xi' \in \Gamma', \rho(\xi') \le |\xi''| < \delta|\xi'|\} \to \mathbb{C}$, i.e., $\mu \in \mathcal{M}^2(\Gamma')$. In this case, we choose a finite collection of open convex cones $\Gamma''_j \subset \mathbb{R}^{n-d}$ with $\cup_{j=1}^s \Gamma''_j = \dot{\mathbb{R}}^{n-d}$ and write μ in the form $\mu = \sum_{j=1}^s \mu_j$ with the μ_j defined on $\{\xi \in \mathbb{R}^n; \xi' \in \Gamma', \rho(\xi') \le |\xi''| < \delta|\xi'|\} \to \mathbb{C}$, but supported in $\{\xi \in \mathbb{R}^n; \xi' \in \Gamma', \xi'' \in \Gamma''_j, \rho(\xi') \le |\xi''| < \delta|\xi'|\} \to \mathbb{C}$ and satisfying $|\mu_j(\xi)| \le |\mu(\xi)|$. We have then already given a meaning to $\mathcal{F}^{-1}(\mu_j)$ and set

$$\mathcal{F}^{-1}_{\Gamma' \times \mathbb{R}^{n-d}}(\mu) = \sum_{j=1}^{s} \mathcal{F}^{-1}_{\Gamma' \times \Gamma''_j}(\mu_j). \tag{7.5}$$

It is easy to see that as a second microfunction (i.e., as an element in $\mathcal{B}^2/\mathcal{A}^2$) on $\mathbb{R}^d \times \Gamma' \times \mathbb{R}^{n-d}$, $\mathcal{F}^{-1}(\mu)$ does not depend on the splitting of μ in the form $\mu = \sum_{j=1}^s \mu_j$.

Remark 7.2. *Assume that $\mu \in \mathcal{M}^2(\Gamma')$ and that some open (not necessarily convex) cone $\Gamma'' \subset \dot{\mathbb{R}}^{n-d}$ is given such that $\mu \sim 0$ in $\mathcal{M}^2(\Gamma', \Gamma'')$. Then*

$$\operatorname{supp} \mathcal{F}^{-1}(\mu) \subset \mathbb{R}^d \times \overline{\Gamma'} \times \mathbb{R}^{n-d} \times (\dot{\mathbb{R}}^{n-d} \setminus \Gamma''). \tag{7.6}$$

Remark 7.3. *The main reason why in (7.4) we restrict integration to the set $\rho(\xi') \le |\xi''| \le \delta_j|\xi'|$ is that (7.2) is only known to hold there. Let us in fact for later use discuss the following situation: we are given some measurable function $\nu : \Gamma' \times \Gamma'' \to \mathbb{C}$ such that*

$$|\nu(\xi)| \le \exp\left[\ell(\xi') + |\xi''|/j\right], \ \text{for } \xi \in \Gamma' \times \Gamma''. \tag{7.7}$$

Also define h', h'', by

$$h'(z) = \int_{\Gamma' \times \Gamma'', \delta_j|\xi'| \le |\xi''|} \exp\left[i\langle z, \xi\rangle\right] \nu(\xi)\, d\xi, \tag{7.8}$$

$$h''(z) = \int_{\Gamma' \times \Gamma'', |\xi''| \le \rho'(\xi')} \exp\left[i\langle z, \xi\rangle\right] \nu(\xi)\, d\xi. \tag{7.9}$$

Then $b_{\operatorname{Im} z' \to 0}(h')(\cdot, z'')$ vanishes as a microfunction with holomorphic parameters defined on a set of the form $\mathbb{R}^d \times \dot{\mathbb{R}}^d \times (\mathbb{R}^{n-d} + i(\dot{y}''/j + G''))$ with $\dot{y}'' \in G'' = \operatorname{Int} \Gamma''^{\perp}$. On the other hand, $b_{\operatorname{Im} z' \to 0}(h'')(\cdot, z'')$ does not in general vanish as a microfunction with holomorphic parameters. However, it is defined globally with respect to the z''-variable, i.e., $b_{\operatorname{Im} z' \to 0}(h'')(\cdot, z'') \in \mathcal{CO}_N(\mathbb{R}^d \times \dot{\mathbb{R}}^d \times \mathbb{C}^{n-d})$, and gives a $\mathcal{A}^2_\Sigma$ contribution. This shows that $\mathcal{F}^{-1}(\nu)$ does not depend on the choice of ρ in the regularization procedure, and also implies that when μ is a measurable function which satisfies the estimate (7.7), then we can define $\mathcal{F}^{-1}(\nu)$ as a second hyperfunction rather than only as a second microfunction. See the explanation after proposition 7.4.

The results in the following proposition are obvious.

Proposition 7.4. *a) Assume that $\mu \in \mathcal{M}^2(\Gamma')$ and that it satisfies for some sublinear function ℓ and some constants $c, d > 0$ the estimate*

$$|\mu(\xi)| \le \exp[\ell(\xi') - d|\xi''|], \xi' \in \Gamma', |\xi''| \le c|\xi'|, |\xi''| > \rho(\xi'). \tag{7.10}$$

Then $\mathcal{F}^{-1}(\mu) = 0$. In particular, equivalent μ's will lead to the same Fourier-inverse transform.

b) Assume that there is an open convex cone $\Gamma'' \subset \dot{\mathbb{R}}^{n-d}$ so that

$$|\mu(\xi)| \le \exp[\ell(\xi') - d|\xi''|], \xi' \in \Gamma', \xi'' \in \Gamma'', |\xi''| \le c|\xi'|, |\xi''| > \rho(\xi'). \tag{7.11}$$

Then $\mathcal{F}^{-1}(\mu) = 0$ as a second microfunction on $\mathbb{R}^d \times \mathbb{R}^{n-d} \times \Gamma' \times \Gamma''$.

As remarked already a number of times, calculation of $\mathcal{F}^{-1}(\mu)$ is considered here in second microfunctions rather than second hyperfunctions. When $\mu \in \mathcal{M}(\Gamma')$, this is due to the fact that estimates are know to hold in general only for $|\xi''| \ge \rho(\xi')$. When on the other hand, μ is defined on a set of form $\{\xi \in \mathbb{R}^n; \xi' \in \Gamma', |\xi''| \le \delta|\xi'|\}$ and satisfies $|\mu(\xi)| \le \exp[\ell(\xi') + \varphi(\xi)]$ on that set, then we can define a regularization of the formal integral $\int \exp[i\langle x, \xi\rangle]\mu(\xi)\, d\xi$ in second hyperfunctions on $\mathbb{R}^n \times \Gamma'$ in the following way: we consider a finite collection of open convex cones $\Gamma''_k \subset \mathbb{R}^{n-d}$ with $\cup_k \Gamma''_k = \dot{\mathbb{R}}^{n-d}$, and split μ into a sum of form $\mu = \sum_k \mu_k$ with $\mu_k(\xi) = 0$ if $\xi \notin \Gamma' \times \bar{\Gamma}''_k$, $|\mu_k(\xi)| \le |\mu(\xi)|$. We are then left with the problem of regularizing $\mathcal{F}^{-1}(\mu_k)$ in second hyperfunctions, and this will be done by considering regular sequences of form $h_{kj}(z) = \int_{|\xi''| \le \delta_j |\xi'|} \exp[i\langle x, \xi\rangle]\mu_k(\xi)\, d\xi$. The main new thing is here of course that we do not cut away the part $|\xi''| \le \rho(\xi')$ in the domains of integration. It is easy to see that $\sum_k \mathcal{F}^{-1}(\mu_k)$ is then well-defined as a second hyperfunction.

Also the following remark is elementary

Remark 7.5. *Let μ be a measurable function on $\Gamma = \{\xi \in \mathbb{R}^n; \xi \in \Gamma' \times \mathbb{R}^{n-d}, |\xi''| < \delta|\xi'|\}$, $0 < \delta < \infty$, and assume that for some sublinear function ℓ we have*

$$|\mu(\xi)| \le \exp \ell(\xi'), \forall \xi \in \Gamma. \tag{7.12}$$

This is thus a function in $\mathcal{M}^2(\Gamma')$, but it is also a function of the type for which one can calculate the Fourier-inverse transform in classical hyperfunctions. Indeed, we can calculate the Fourier-inverse $\mathcal{F}^{-1}(\mu)$ in the following way: at first we consider the function

$$h(z) = \int_{\xi \in \Gamma} \exp[i\langle z, \xi\rangle]\mu(\xi)\, d\xi. \tag{7.13}$$

It is immediate that h is defined and analytic on the set $\{z \in \mathbb{C}^n; \operatorname{Im} z \in \operatorname{Int}\Gamma^\perp\}$. We can therefore set $u = (2\pi)^{-n} b(h)$ where $b(h)$ means the hyperfunctional boundary value in first hyperfunctions.

On the other hand, we can also calculate $u' = \mathcal{F}^{-1}(\mu)$ as a second hyperfunction. It is then interesting to note that u' is precisely the second hyperfunction associated with u by the immersion of microfunctions into second hyperfunctions. In fact, we can take a decomposition $\mu = \sum_k \mu_k$ with $\operatorname{supp}\mu_k \subset \Gamma \cap \{\xi'' \in \Gamma''_k\}$ for open convex cones $\Gamma''_k \subset \dot{\mathbb{R}}^{n-d}$ as above, and define

$$h_k(z) = \int_{\xi\in\Gamma,\xi''\in\Gamma''_k} \exp\left[i\langle z,\xi\rangle\right]\mu_k(\xi)\, d\xi.$$

Then h_k is holomorphic on $\mathbb{R}^n + i(\operatorname{Int}\Gamma^\perp + \operatorname{Int}\Gamma''^{\perp}_k)$ and each h_k forms a constant regular sequence on $\mathbb{R}^d \times \dot{\mathbb{R}}^d \times (\mathbb{R}^{n-d} + i\operatorname{Int}\Gamma''^{\perp}_k)$, which we sum up to define u'. It follows then immediately from the definition that these h_k satisfy $h = \sum_k h_k$ on their common domain of definition, which corresponds the decomposition (2.6) in the explanation of the imbedding morphism $\mathcal{C}_M|_\Sigma \to \mathcal{B}^2_\Sigma$ in section 2.

The preceding remark simplifies calculations in many situations, in that we can argue classically if this makes sense. Thus in particular, the Fourier inverse of the function 1 in second microlocalization is just the class of the Delta-distribution in second hyperfunctions.

An interesting situation appears when rather than starting from a function $\mu \in \mathcal{M}^2(\Gamma', \Gamma'')$ we start from a regular sequence $\{\mu_j\}_{j\geq 1}$ defined on $\Gamma' \times \Gamma''$. We can then associate with the sequence some $\mu \in \mathcal{M}^2(\Gamma', \Gamma'')$ as in theorem 6.3 and then calculate $\mathcal{F}^{-1}(\mu)$. It follows immediately that $\mathcal{F}^{-1}(\mu)$ does not depend on the choice of μ, so we can in fact define $\mathcal{F}^{-1}(\{\mu_j\}_j)$ by $\mathcal{F}^{-1}(\{\mu_j\}_j) = \mathcal{F}^{-1}(\mu)$. That this is a natural construct follows from the fact that when we will want to calculate the direct Fourier-transform of some second hyperfunction, we will in fact intuitively have to calculate the Fourier-transforms of the elements h_j of some regular sequence of holomorphic functions, and thus arrive in a first step to some sequence μ_j, $j \geq 1$, rather than to some single function μ. We shall not further develop these ideas in this paper however.

Theorem 7.6. *Let $\Gamma'_1 \subset\subset \Gamma'$, $\Gamma''_1 \subset\subset \Gamma''$, $\delta_{1j} \leq \delta_j$ and consider $\mu \in \mathcal{M}^2(\Gamma', \Gamma'')$. Define B_j by*

$$B_j = \{\xi; \xi \in ((\Gamma' \setminus \Gamma'_1) \times \Gamma) \cup (\Gamma' \times (\Gamma'' \setminus \Gamma''_1))\} \cup \{\xi \in \Gamma' \times \Gamma'', \delta_{1j}|\xi'| \leq |\xi''| \leq \delta_j|\xi'|\}$$

and h_j by

$$h_j(z) = \int_{B_j, |\xi''|\leq\delta_j|\xi'|} \exp\left[i\langle z,\xi\rangle\right]\mu(\xi)\, d\xi. \tag{7.14}$$

Then h_j is a regular sequence and $b^2(\{h_j\}_{j\geq 1}) = 0$ on $\mathbb{R}^n \times \operatorname{Int}\Gamma'_1 \times \operatorname{Int}\Gamma''_1$.

Proof. The first statement, about "regular sequences" is by now trivial. For the remaining part of the statement, there are three different situations to be considered: i) when $\operatorname{supp}\mu \subset (\Gamma' \setminus \Gamma'_1) \times \Gamma)$, ii) when $\operatorname{supp}\mu \subset (\Gamma' \times (\Gamma'' \setminus \Gamma''_1))$, and iii) when $\operatorname{supp}\mu \subset \{\xi \in \Gamma' \times \Gamma'', \delta_{1j}|\xi'| \le |\xi''| \le \delta_j|\xi'|\}$.

In the case i) the single h_j vanish as microfunctions on $\mathbb{R}^d \times \Gamma'$ with holomorphic parameter z'' in $\mathbb{R}^{n-d} + i\Gamma''^{\perp}$. Therefore $b^2(\{h_j\}_{j\ge1}) = 0$ in this case.

In case ii), we can write μ as a finite sum of functions μ' in $\mathcal{M}^2(\Gamma' \times \Gamma'')$ which have supports in convex cones of form $\Gamma' \times \tilde{\Gamma}''$ contained in $\Gamma' \times (\Gamma'' \setminus \Gamma''_1)$. We may then from the very beginning assume that μ has this property. Denote in fact by $\tilde{\Gamma}''$ a closed convex cone such that $\tilde{\Gamma}'' \subset (\Gamma'' \setminus \Gamma''_1))$ and assume that $\operatorname{supp}\mu$ is contained in $\Gamma' \times \tilde{\Gamma}''$. The h_j are then a regular sequence of holomorphic functions defined on $\mathbb{R}^n + i(\Gamma'^{\perp} \times \tilde{\Gamma}''^{\perp})$. The corresponding microfunction with holomorphic parameter z'' is thus defined on $\mathbb{R}^d \times \Gamma' \times (\mathbb{R}^{n-d} + i\tilde{\Gamma}''^{\perp})$. Therefore $b^2(\{h_j\}_j) = 0$ on $\mathbb{R}^n \times \operatorname{Int}\ \Gamma'_1 \times \Gamma''_1$.

Finally note that the case iii) is trivial, in that then the boundary values of the corresponding h_j already vanish as microfunctions on $\mathbb{R}^d \times \dot{\mathbb{R}}^d$ with holomorphic parameter z'' in $\mathbb{R}^{n-d} + i\Gamma''^{\perp}$ □

In the sequel a function $\mu : F \to \mathbb{C}$, F open in $\mathbb{C}^n$, shall be called almost analytic, if there is a constant $d > 0$ and a sublinear ℓ, so that

$$|\bar{\partial}\mu(\zeta)| \le \exp\left[\ell(\operatorname{Re}\zeta') - d|\operatorname{Re}\zeta''|\right] \quad \text{for } \zeta \in F. \tag{7.15}$$

The main result in this section is now the following result of Paley-Wiener type (for a related result in distributions, cf. proposition 2.1.15 in [10]):

Theorem 7.7. *Consider Γ', Γ'', ρ, as above and let $\mu : A \to \mathbb{C}$ be a function with the following properties:*

a) $|\mu(\xi)| \le \exp\left[\ell(\xi') + \varphi(\xi)\right],\ \forall \xi \in A,$

b) there exists a sublinear function ρ and an almost analytic extension of μ to a set of form:

$$F = \{\zeta \in \mathbb{C}^n; |\operatorname{Im}\zeta| < c|\operatorname{Re}\zeta''|, \operatorname{Re}\zeta' \in \Gamma', \operatorname{Re}\zeta'' \in \Gamma'', |\operatorname{Re}\zeta''| < \delta|\operatorname{Re}\zeta'|,$$
$$|\operatorname{Re}\zeta''| > \rho(\operatorname{Re}\zeta')\} \tag{7.16}$$

such that

$$|\mu(\zeta)| \le \exp\left[\ell(\operatorname{Re}\zeta') + \varphi(\operatorname{Re}\zeta) + \varepsilon|\operatorname{Im}\zeta|\right], \quad \textit{on } F. \tag{7.17}$$

Then $\mathcal{F}^{-1}(\mu) = 0$ in $\mathcal{C}^2(|x| > \varepsilon, \xi \in \operatorname{Int}\Gamma' \times \operatorname{Int}\Gamma'')$.

Proof of theorem 7.7. The assumptions in the theorem refer to the set F (defined in (7.16)), but in the proof it is more convenient to assume that we work on the larger but, geometrically speaking, simpler, set

$$F' = \{\zeta \in \mathbb{C}^n; |\operatorname{Im}\zeta| < c|\operatorname{Re}\zeta''|, \operatorname{Re}\zeta' \in \Gamma', \operatorname{Re}\zeta'' \in \Gamma'', |\operatorname{Re}\zeta''| < c|\operatorname{Re}\zeta'|\}. \quad (7.18)$$

This is possible in view of the following remark: if μ is defined on F, satisfies (7.17) and is almost analytic there, then we can replace it by an equivalent μ', which is defined on some set of form F', satisfies (7.17) on F', and is almost analytic on F'. Indeed, it is trivial that, replacing ρ, if necessary, by some larger ρ', we can find an extension μ' of μ defined on F' such that we have (7.17) and for which

$$|\bar{\partial}\mu'(\zeta)| \leq \tilde{c}\exp[\ell(\operatorname{Re}\zeta') + \varphi(\operatorname{Re}\zeta) + \varepsilon|\operatorname{Im}\zeta|], \text{ on } F' \setminus F.$$

The new part in F' is when $|\operatorname{Re}\zeta''| \leq \rho(\operatorname{Re}\zeta')$, (for some suitable sublinear function ρ,) so we should essentially estimate $\varphi(\operatorname{Re}\zeta) + \varepsilon|\operatorname{Im}\zeta|$ by $\ell'(\operatorname{Re}\zeta') - d|\operatorname{Re}\zeta''|$ on such sets, if ℓ' and $d > 0$ are suitable. Here we note that we actually only need this on $|\operatorname{Im}\zeta| < c|\operatorname{Re}\zeta''|$. We now apply the following remarks:

i) on sets of form $|\operatorname{Re}\zeta''| \leq \rho(\operatorname{Re}\zeta')$, $\varphi(\operatorname{Re}\zeta)$ can be majorized by $\ell''(\operatorname{Re}\zeta')$ for some sublinear function ℓ'' and thus $d|\operatorname{Re}\zeta''| \leq d\rho(\operatorname{Re}\zeta')$,

ii) we can estimate $\varepsilon|\operatorname{Im}\zeta|$ by $c|\operatorname{Re}\zeta''|$, which in turn is etimated by $c\rho(\operatorname{Re}\zeta')$.

Therefore $\varphi(\operatorname{Re}\zeta) + \varepsilon|\operatorname{Im}\zeta| \leq \ell''(\operatorname{Re}\zeta') + c\rho(\operatorname{Re}\zeta') + d|\operatorname{Re}\zeta''| - d|\operatorname{Re}\zeta''| \leq \ell''(\operatorname{Re}\zeta') + c\rho(\operatorname{Re}\zeta') + d\rho(\operatorname{Re}\zeta') - d|\operatorname{Re}\zeta''|$.

We now return to the proof of theorem 7.7, assuming that μ is defined and almost analytic on F' and satisfies (7.17) with F replaced by F'. We have to show that the regular sequence

$$h_j(z) = \int_{\xi' \in \Gamma', \xi'' \in \Gamma'', |\xi''| \leq \delta_j|\xi'|} \exp[i\langle z, \xi\rangle]\mu(\xi)\, d\xi$$

gives the zero- second microfunction on $\{|x| > \varepsilon\} \times \operatorname{Int}\Gamma' \times \operatorname{Int}\Gamma''$. Let us then fix x^0 with $|x^0| > \varepsilon$. We want to show that $\mathcal{F}^{-1}(\mu) = 0$ near $x^0 \times \operatorname{Int}\Gamma' \times \operatorname{Int}\Gamma''$. We shall do this by writing $\{h_j\}_j$ as a sum of three regular sequences, each of which with appropriate analytic extensions. The basic idea is to deform the integration contour $\{\xi; \xi' \in \Gamma', \xi'' \in \Gamma''; |\xi''| \leq \delta_j|\xi'|\}$ into the complex domain. More precisely, we shall consider the set

$$B = \{\zeta \in \mathbb{C}^n; \zeta = \xi + i\sigma\frac{x^0}{|x^0|}|\xi''|, \xi' \in \Gamma', \xi'' \in \Gamma'', |\xi''| \leq \delta_j|\xi'|, 0 \leq \sigma \leq \delta\}. \quad (7.19)$$

We note then that $\partial B = D_1 \cup D_2 \cup D_3$, where

$$D_1 = \{\xi; \xi' \in \Gamma', \xi'' \in \Gamma'', |\xi''| \leq \delta_j|\xi'|\}, D_2 = \{\zeta \in \mathbb{C}^n; \zeta = \xi + i\delta\frac{x^0}{|x^0|}|\xi''|, \xi \in D_1\},$$

$$D_3 = \{\zeta \in \mathbb{C}^n; \zeta = \xi + i\sigma\frac{x^0}{|x^0|}|\xi''|, \xi \in \partial D_1, 0 \le \sigma \le \delta\}.$$

At this moment we basically want to apply the Stokes theorem, in order to replace integration on D_1 by integration on B, D_2 and D_3. Since we want to use the assumption on $\bar\partial\mu$, it is natural to write $d\xi$ on $\mathbb{R}^n$ as $d\zeta_1 \wedge \cdots \wedge d\zeta_n$. The Stokes theorem then states that

$$\int_{\partial B} \exp[i\langle z, \zeta\rangle]\mu(\zeta) d\zeta_1 \wedge \cdots \wedge d\zeta_n = \int_B \exp[i\langle z, \zeta\rangle] \sum_{j=1}^{n} \frac{\partial}{\partial\bar\zeta_j}\mu(\zeta) d\bar\zeta_j \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

which, by abuse of notation, we shall write as

$$\int_{\partial B} \exp[i\langle z, \zeta\rangle]\mu(\zeta) d\zeta = \int_B \exp[i\langle z, \zeta\rangle]\bar\partial\mu(\zeta)\, d\lambda(\zeta).$$

It follows that we have

$$h_j(z) = I_j(z) + II_j(z) + III_j(z)$$

where

$$I_j(z) = \int_B \exp[i\langle z, \zeta\rangle]\, \bar\partial\mu(\zeta)\, d\lambda(\zeta)\ , II_j(z) = -\int_{D_2} \exp[i\langle z, \zeta\rangle]\mu(\zeta)\, d\zeta\ ,$$

$$III_j(z) = -\int_{D_3} \exp[i\langle z, \zeta\rangle]\mu(\zeta)\, d\zeta.$$

What we have to show is that all three integrals define regular sequences of holomorphic functions which can be extended analytically to appropriate sets in such a way that their second boundary value vanishes.

In the case of $I_j(z)$ this follows from our assumption on $\bar\partial\mu$. Indeed, by assumption $|\bar\partial\mu(\xi + i\sigma\frac{x^0}{|x^0|}|\xi''|)|$ can be estimated by

$$\tilde c \exp[\ell(\mathrm{Re}(\xi + i\sigma\frac{x^0}{|x^0|}|\xi''|)') - d|\,\mathrm{Re}(\xi + i\sigma\frac{x^0}{|x^0|}|\xi''|)''|] = \tilde c \exp[\ell(\xi') - d|\xi''|],$$

whereas (with the notation ζ for $\xi + i\sigma\frac{x^0}{|x^0|}|\xi''|$)

$$\begin{aligned}\mathrm{Re}\, i\langle z, \xi + i\sigma\frac{x^0}{|x^0|}|\xi''|\rangle &\le -\langle x^0, \mathrm{Im}\,\zeta\rangle + |\,\mathrm{Re}\, z - x^0||\,\mathrm{Im}\,\zeta| - \langle \mathrm{Im}\, z, \mathrm{Re}\,\zeta\rangle \\ &\le (\delta|\,\mathrm{Re}\, z - x^0| + |\,\mathrm{Im}\, z''|)|\xi''| - \langle \mathrm{Im}\, z', \xi'\rangle.\end{aligned}$$

It is clear from this that for z with $\mathrm{Im}\, z' \in \mathrm{Int}\,\Gamma'^{\perp}$ and $\delta|\,\mathrm{Re}\, z - x^0| + |\,\mathrm{Im}\, z''| < d/2$, our integrand can be estimated by $\tilde c \exp[\ell(\xi') - \langle \mathrm{Im}\, z', \xi'\rangle - (d/2)|\xi''|]$ which is of exponential-decay type, so $I_j(z)$ is holomorphic on

$$\{z \in \mathbb{C}^n; \mathrm{Im}\, z' \in \mathrm{Int}\,\Gamma'^{\perp}, \delta|\,\mathrm{Re}\, z - x^0| < d/4, |\,\mathrm{Im}\, z''| < d/4\},$$

and gives $\mathcal{A}^2_\Sigma$ contribution, that is, zero contribution as a second microfunction.

We next study II_j. For $\zeta \in D_2$ we have

$$\begin{aligned}\operatorname{Re} i\langle z,\zeta\rangle = \operatorname{Re} i\langle z, \xi + i\delta\frac{x^0}{|x^0|}|\xi''|\rangle &\leq -\delta|x^0||\xi''| + \delta|\operatorname{Re} z - x^0|\,|\xi''| - \langle \operatorname{Im} z, \xi\rangle \\ &\leq -(\delta|x^0| - \delta|\operatorname{Re} z - x^0| - |\operatorname{Im} z''|)|\xi''| - \langle \operatorname{Im} z', \xi'\rangle\end{aligned}$$

and

$$|\mu(\xi + i\delta\frac{x^0}{|x^0|}|\xi''|)| \leq \exp\left[\ell(\xi') + (\varepsilon\delta + 1/j)|\xi''|\right].$$

Since $|x^0| > \varepsilon$ we get again exponential decay in $|\xi'|$ and $|\xi''|$, if $\operatorname{Im} z' \in \operatorname{Int}\Gamma'^{\perp}$ and $\delta|\operatorname{Re} z - x^0| + |\operatorname{Im} z''| < (\delta|x^0| - \delta\varepsilon - 1/j)/2$ for large enough j. The argument is then continued as in the case of I_j.

Finally, we have to study III_j. The boundary of D_1 has the following structure:

$$\begin{aligned}\partial D_1 = (\{\xi; \xi \in \partial\Gamma' \times \Gamma'', |\xi''| \leq \delta_j|\xi'|\}) &\cup (\{\xi; \xi \in \Gamma' \times \partial\Gamma'', |\xi''| \leq \delta_j|\xi'|\}) \\ &\cup \{\xi \in \Gamma' \times \Gamma''; |\xi''| = \delta_j|\xi'|\}.\end{aligned}$$

The relevant estimate for μ is

$$|\mu(\xi + i\sigma\frac{x^0}{|x^0|}|\xi''|)| \leq \exp\left[\ell(\xi') + (\varepsilon\sigma + 1/j)|\xi''|\right]. \tag{7.20}$$

As for $\operatorname{Re} i\langle z,\zeta\rangle$, $\zeta \in D_3$, we notice that

$$-\langle \operatorname{Re} z, \operatorname{Im}\zeta\rangle = -\langle \operatorname{Re} z, \sigma\frac{x^0}{|x^0|}|\xi''|\rangle = -\sigma|x^0|\,|\xi''| + \sigma|\operatorname{Re} z - x^0|\,|\xi''|.$$

Again this will suffice to compensate for the term $\varepsilon\sigma|\xi''|$ in the exponential of the right hand-side of (7.20) if $|x^0| > \varepsilon$ and $|\operatorname{Re} z - x^0|$ is small. As for $-\langle \operatorname{Im} z, \operatorname{Re}\zeta\rangle = -\langle \operatorname{Im} z, \xi\rangle$, we will have to consider the cases $\xi \in \partial\Gamma' \times \Gamma''$, $\xi \in \Gamma' \times \partial\Gamma''$, and $\{\xi; \xi \in \Gamma' \times \Gamma'', |\xi''| = \delta_j|\xi'|\}$ separately. However, we are here in a situation analoguous to the one considered in theorem 7.6 and we can omit details. □

8 Fourier-inverses and derivatives

We study in this section derivatives of Fourier-inverses in second microlocalization. The assumption is here that $\mu \in \mathcal{M}^2(\Gamma', \Gamma'')$. Denote, as in section 7, by A sets of form $\{\xi; \xi' \in \Gamma', \xi'' \in \Gamma'', |\xi''| < \delta|\xi'|, |\xi''| \geq \rho(\xi')\}$.

Then, if A is suitable,

$$\frac{\partial h_j(z)}{\partial z_k} = \int_{\xi \in A, |\xi''| \leq \delta_j|\xi'|} \exp\left[i\langle z,\xi\rangle\right](i\xi_k)\mu(\xi)\,d\xi. \tag{8.1}$$

This shows that the classical formula $\mathcal{F}^{-1}(\xi_k\mu) = (-i\partial/\partial x_k)\mathcal{F}^{-1}(\mu)$ holds also in second microfunctions.

Only slightly more complicated is the case when we consider multiplication of $\mathcal{F}^{-1}(\mu)$ with x_k, for some fixed $k \in \{1, \ldots, n\}$. We must assume here of course that μ can be derivated in ξ_k and that $(\partial/\partial\xi_k)\mu \in \mathcal{M}^2(\Gamma', \Gamma'')$. We use that if $u = \mathcal{F}^{-1}(\mu)$ is the second microfunction associated with the regular sequence $j \to h_j$, then $x_k u$ is the second microfunction associated with the regular sequence $j \to z_k h_j(z)$ and that $z_k \exp[i\langle z, \xi\rangle] = -i(\partial/\partial\xi_k)\exp[i\langle z, \xi\rangle]$. Let us also choose a sequence of C^∞- functions $j \to e_j$ as in section 5. Then we may assume that h_j is given by

$$h_j(z) = \int_{\xi\in A, |\xi''|\le\delta_j|\xi'|} \exp[i\langle z, \zeta\rangle]\, e_j(\xi)\mu(\xi)\, d\xi.$$

It follows in particular that

$$iz_k h_j(z) = \int_{\xi\in A, |\xi''|\le\delta_j|\xi'|} \{(\partial/\partial\xi_k)\exp[i\langle z, \xi\rangle]\}\, e_j(\xi)\mu(\xi)\, d\xi = \\ -\int_{\xi\in A, |\xi''|\le\delta_j|\xi'|} \exp[i\langle z, \xi\rangle](\partial/\partial\xi_k)[e_j(\xi)\mu(\xi)]\, d\xi. \qquad (8.2)$$

(It is now also clear, why we introduced the cut-off functions e_j: by using them we have avoided boundary terms from the partial integrations in the last equality, in that e_j vanishes of infinite order on the boundary of $\{\xi \in A; |\xi''| \le \delta_j|\xi'|\}$. Such boundary terms could have been treated in principle also quite easily, but they lead to Fourier-inverses of measures on the boundary of the sets $\{\xi \in A, |\xi''| \le \delta_j|\xi'|\}$ with growth-type conditions rather than of measurable functions in $\mathcal{M}^2(\Gamma', \Gamma'')$.)

Here, however, the regular sequences

$$h_j'(z) = \int_{\xi\in A, |\xi''|\le\delta_j|\xi'|} \exp[i\langle z, \xi\rangle](\partial/\partial\xi_k)[e_j(\xi)\mu(\xi)]\, d\xi$$

and

$$h_j''(z) = \int_{\xi\in A, |\xi''|\le\delta_j|\xi'|} \exp[i\langle z, \xi\rangle](\partial/\partial\xi_k)\mu(\xi)\, d\xi$$

define the same second microfunction. We obtain, all in all, that

$$-ix_k\mathcal{F}^{-1}(\mu) = \mathcal{F}^{-1}((\partial/\partial\xi_k)\mu).$$

It is easy to see that the same holds also for second hyperfunctions associated with functions in $\mathcal{M}^2(\Gamma')$ and we omit details.

9 $\mathcal{B}^2$-solutions for Turrittin's equations

At first we describe an abstract setting in which our calculations can be performed. We start from some natural number k and some rational number $0 < \alpha < 1$ and denote by $a = \alpha/(1+\alpha)$. Thus, α and a are somehow conjugate in the sense that $1/\alpha - 1/a = 1$. The assumptions on α imply $a > 0$. We also choose some constants $a_{jrs} \in \mathbb{C}$, j, r, s natural numbers, such that $a_{jrs} = 0$ unless

$$r < k, r + s \leq k, \alpha(k - r + j) = s. \tag{9.1}$$

Note that this gives a bound also for j, so there are only finitely many constants $a_{jrs} \neq 0$. We next consider the operator in two variables

$$p(x, D) = i^{-k} \frac{\partial^k}{\partial x_1^k} u + \sum_{j,r,s} i^{j-r-s} a_{jrs} \frac{\partial^r}{\partial x_1^r} x_1^j \frac{\partial^s}{\partial x_2^s} \ , i = \sqrt{-1}. \tag{9.2}$$

The fact that multiplication with x_1^j comes after derivation in x_1, is of course for later convenience.

By making a formal Fourier transform we obtain the operator

$$q(\xi, \partial_\xi) = \xi_1^k + \sum_{j,r,s} a_{jrs} \xi_1^r \xi_2^s \frac{\partial^j}{\partial \xi_1^j}. \tag{9.3}$$

This is actually an ordinary differential operator in ξ_1 with ξ_2 as a parameter. We are interested in a solution of the equation $q(\xi, \partial_\xi)\varphi = 1$. The formal Fourier-inverse of such a φ would then be a solution of $p(x, D)u = \delta$, δ the Dirac distribution. (See 7.5 and section 8.) We shall see in a moment that in some interesting situations, we can give a meaning to this in second hyperfunctions.

To study $q(\xi, \partial_\xi)\varphi = 1$, we shall denote by $Q(t, d/dt)$ the operator

$$Q(t, d/dt) = t^k + \sum_{j,r,s} a_{jrs} t^r \frac{d^j}{dt^j}, \tag{9.4}$$

and denote by u a function such that $Q(t, d/dt)u(t) = 1$. If we denote by μ the function $\mu(\xi) = u(\xi_1/\xi_2^\alpha)$, then we shall have

$$q(\xi, \partial_\xi)\mu(\xi) = \xi_1^k u(\xi_1/\xi_2^\alpha) + \sum_{j,r,s} a_{jrs} \xi_1^r \xi_2^{s-j\alpha} \left(\frac{d^j}{dt^j} u\right) (\xi_1/\xi_2^\alpha) =$$

$$\xi_2^{k\alpha}[(\xi_1/\xi_2^\alpha)^k u(\xi_1/\xi_2^\alpha) + \sum_{j,r,s} a_{jrs} \xi_2^{s-k\alpha+r\alpha-j\alpha} (\xi_1/\xi_2^\alpha)^r \left(\frac{d^j}{dt^j} u\right) (\xi_1/\xi_2^\alpha)] = \xi_2^{k\alpha},$$

since we assumed that $s - k\alpha + r\alpha - j\alpha = 0$ for all j, r, s such that $a_{jrs} \neq 0$. The sought-for solution for $q(\xi, \partial_\xi)\varphi(\xi) = 1$ is then $\varphi(\xi) = \mu(\xi)/\xi_2^{k\alpha}$. To obtain reasonable estimates for φ, we shall now assume in addition that the equation $Q(t, dt)u(t) = 1$ is known to admit a solution u such that $u(t) = O(\exp[|t|^{1+a}])$ for the constant a introduced above. Since $\alpha(1+a) = a$, this gives for φ that

$$|\varphi(\xi)| \le c|\xi_2|^{-k\alpha} \exp[|\xi_1|^{1+a}/|\xi_2|^a].$$

This means that φ satisfies the estimates of a function of the type considered in section 7 and therefore its Fourier-inverse can be calculated in second hyperfunctions.

Remark 9.1. *To get a feeling of condition (9.1), assume that $a_{j^0,0,k} \neq 0$ for some j^0. Then we must have $\alpha = k/(k+j^0)$ and $a_{j,0,k} = 0$ for all $j \neq j^0$. The "a" corresponding to this α is of course $a = k/j^0$. Assume now also that $a_{jrs} = 0$ whenever $r \neq 0$. The operator Q then reduces to*

$$Q(t, d/dt) = t^k + \sum_{j,0,s} a_{j0s} \frac{d^j}{dt^j}. \tag{9.5}$$

The associated characteristic polynomial is $\lambda \to Q(t,\lambda) = t^k + \sum_{j,0,s} a_{j0s}\lambda^j$ and the roots $\lambda(t)$ of $Q(t,\lambda) = 0$ satisfy $|\lambda(t)| \le c(1+|t|)^{k/j^0}$. WKB-analysis for the solutions of $Q(t, d/dt)u = 0$ therefore indicates that the asymptotic behaviour of these solutions should be of growth order $\exp[c|t|^{1+k/j^0}]$, *which is exactly what we need to make our arguments work.*

We now specialize to the case when $k = 2$ and assume that $a_{jrs} = 0$ when $r \neq 0$. The operators which still fall in this class are then of form

$$\frac{\partial^2}{\partial x_1^2} u + \sum_{j,s} i^{j-s} a_{j0s} x_1^j \frac{\partial^s}{\partial x_2^s}. \tag{9.6}$$

The condition (9.1) comes here to $s \le 2$, $\alpha(2+j) = s$. If $a_{j^002} \neq 2$ for some j^0, then $\alpha = 2/(2+j^0)$, so the only other nonvanishing coefficients a_{j0s} can be a_{j01} for $j = (2+j^0)/2 - 2 = (j^0/2) - 1$. In particular j^0 must be even. Setting $j^0 = 2\ell$, the operators under consideration are thus of form:

$$p(x, D) = (\partial/\partial x_1)^2 + i^{2\ell-2}\gamma x_1^{2\ell}(\partial/\partial x_2)^2 + i^{\ell-2}\lambda x_1^{\ell-1}(\partial/\partial x_2) \tag{9.7}$$

for some natural number $\ell \ge 1$ and some constants γ, λ, $\gamma \neq 0$. We have of course $a = \alpha/(1-\alpha) = (1+\ell)/(1+\ell)\ell = \ell$.

Such operators are particular cases of a class of operators introduced by Grushin. It is known that when $\gamma = 1/i^{(2\ell-2)}$, $p(x, D)$ is solvable classically unless $\lambda = \ell - 2n(\ell+1)$ or $\lambda = \ell - 2 - 2n(\ell+1)$ (n is an arbitrary natural number. Cf. [1]. Strictly speaking, the paper [1] studies distribution solutions, but, since it is known that the classical

Mizohata operator is not locally solvable in hyperfunctions, the argument of [1] also works in hyperfunctions, and shows that the equation (9.7) is not locally solvable in hyperfunctions for the exceptional values of λ mentioned before.) The following discussion is therefore interesting precisely for these exceptional values of λ.

We shall now study the solvability of $p(x, D)E = \delta$ in second hyperfunctions, checking that the operator satisfies the remaining condition considered in the first part of this section. (See later.) The ordinary differential operator Q corresponding to $q(\xi, \partial_\xi)$ is after a renotation for constants

$$Q(t, d/dt) = \left[\left(\frac{d}{dt}\right)^{2\ell} - b\left(\frac{d}{dt}\right)^{\ell-1} - ct^2\right].$$

We have then to study $Q(t, d/dt)u = 1$ and we shall add to this the initial conditions $u(0) = u'(0) = \cdots = u^{2\ell-1}(0) = 0$. The considerations above can then be applied (and this is what we called a moment before "the remaining condition"), if we can show that $|u(t)| \leq c\exp[At^{1+1/\ell}]$. We shall explicitly solve the equation $Q(t, d/dt)u = 1$, together with the initial conditions, by power-series expansion. In fact, assuming that $u = \sum_{j=0}^{\infty} u_j t^j = \sum_{j=2\ell}^{\infty} u_j t^j$, we obtain, with the convention $u_{-2} = u_{-1} = 0$, the following recursive relations for the u_j, for $j \geq 2\ell$:

$$\begin{aligned} u_j &= b\,\frac{(j-\ell-1)\cdots(j-2\ell+1)}{j(j-1)\cdots(j-2\ell+1)}u_{j-\ell-1} + c\,\frac{1}{j(j-1)\cdots(j-2\ell+1)}u_{j-2\ell-2} = \\ &= b\,\frac{1}{j(j-1)\cdots(j-\ell)}u_{j-\ell-1} + c\,\frac{1}{j(j-1)\cdots(j-2\ell+1)}u_{j-2\ell-2}. \end{aligned}$$

We can consequently estimate u_j by

$$|u_j| \leq d\left[\frac{1}{j(j-1)\cdots(j-\ell)}|u_{j-\ell-1}| + \frac{1}{j(j-1)\cdots(j-2\ell+1)}|u_{j-2\ell-2}|\right], \tag{9.8}$$

where $d = \max(|b|, |c|)$.

We shall now prove:

Proposition 9.2. *There are constants c', A, such that*

$$|u(t)| \leq c' \exp[A\,t^{1+1/\ell}],\ t \in \mathbb{R}. \tag{9.9}$$

Proof of proposition 9.2. We may assume that $t \geq 1$ and that $A \geq 1$. Since $\exp[At^{1+1/\ell}] = \sum_{r=0}^{\infty}(A^r t^{r+r/\ell})/r!)$, the proposition follows intuitively if we can show that $|u_j t^j| \leq cA^r\ t^{r+r/\ell}/r!$ for $j = r + r/\ell$, i.e. for $r = \ell j/(\ell+1)$. This is not exactly correct, since r will not in general be entire for entire j. The first thing is of course to work with the Gamma function rather than with factorials. We shall in fact at first show by induction that for some A'

$$|u_j| \leq \frac{A'^j}{\Gamma(j\ell/(\ell+1))}. \tag{9.10}$$

We are therefore reduced to prove that

$$d\frac{\Gamma(\frac{j\ell}{\ell+1})}{\Gamma(\frac{\ell(j-\ell-1)}{\ell+1})}\frac{A'^{-\ell-1}}{j(j-1)\cdots(j-\ell)}+d\frac{\Gamma(\frac{j\ell}{\ell+1})}{\Gamma(\frac{\ell(j-2\ell-2)}{\ell+1})}\frac{A'^{-2\ell-2}}{j(j-1)\cdots(j-2\ell+1)}\leq 1. \tag{9.11}$$

Here we shall assume $\ell \geq 1$ and we may assume that $A' \geq 2d, A' \geq 1$, so it will suffice to show that we have the following two inequalities:

$$\frac{\Gamma(\frac{\ell j}{\ell+1})}{\Gamma(\frac{\ell(j-\ell-1)}{\ell+1})}\frac{1}{j(j-1)\cdots(j-\ell)}\leq 1,$$

$$\frac{\Gamma(\frac{\ell j}{\ell+1})}{\Gamma(\frac{\ell(j-2\ell-2)}{\ell+1})}\frac{1}{j(j-1)\cdots(j-2\ell+1)}\leq 1.$$

Both these inequalities are obvious.

We have now proved (9.10) and want to show that it gives (9.9). In doing so, we may group the terms in $\sum_0^\infty u_j t^j$ into $\ell+1$ sums, acording to the value of j modulo $\ell+1$. More precisely, we write

$$\sum_{j=0}^\infty u_j t^j = \sum_{i=0}^\infty u_{i(\ell+1)} t^{i(\ell+1)} + \sum_{i=0}^\infty u_{i(\ell+1)+1} t^{i(\ell+1)+1} + \cdots + \sum_{i=0}^\infty u_{i(\ell+1)+\ell} t^{i(\ell+1)+\ell}. \tag{9.12}$$

We can now estimate the single sums and take the case of $\sum_{i=0}^\infty u_{i(\ell+1)+1} t^{i(\ell+1)+1}$ for exemplification. We can estimate this by

$$\sum_{i=0}^\infty |u_{i(\ell+1)+1}|\,|t|^{i(\ell+1)+1} \leq |A't| \sum_{i=0}^\infty \frac{|A't|^{i(\ell+1)}}{\Gamma(i\ell+\ell/(\ell+1))}$$
$$\leq |A't| \sum_{i=0}^\infty \frac{(|A't|^{1+1/\ell})^{i\ell}}{(i\ell-1)!} \leq |A't|^{2+1/\ell} \exp\,[|A't|^{1+1/\ell}].$$

It suffices now to fix A and $c' > 0$ with $A > A'^{1+1/\ell}$ and such that $|A't|^{2+1/\ell} < c' \exp[(A - A'^{1+1/\ell})|t|^{1+1/\ell}]$. □

Similar arguments are valid for many other equations. We shall in fact now consider a second class of equations of the type of the equations considered in the beginning of this section, namely equations of form

$$T_{k,m}u = [\frac{\partial^k}{\partial x_2^k} + a x_2^m \frac{\partial^k}{\partial x_1^k}]u = f, \tag{9.13}$$

where k and m are natural numbers. When $k = 1$ and $a = i$ we thus just obtain Mizohata's equations. (Traditionally the Mizohata operators are written as $u \to (\partial/\partial x_1 + ix_1^m \partial/\partial x_2)u$, but in the present context we prefer to interchange the roles of x_1 and x_2.) In the case of general m and k we shall call $T_{m,k}u = f$ "Turrittin's equations". The reason for this terminology will be explained below. The only non-vanishing term is with $j = m, r = 0, s = k$, so the condition (9.1) holds with $\alpha = k/(k+m)$. The ordinary differential operator $Q(t, d/dt)$ associated with p is

$$Q(t, d/dt) = [\frac{d^m}{d\,t^m} + t^k]. \tag{9.14}$$

The value $a = \alpha/(1-\alpha)$ associated with α is $a = k/m$. The fact that the arguments introduced above are applicable for the present situation follows then from the following

Lemma 9.3. *Let u be a solution of $Q(t, d/dt)u = 1$ with the initial conditions: $u(0) = \cdots = u^{(m-1)} = 0$. Then there are constants c, c' so that*

$$|u(t)| \leq c \exp{[ct^{1+k/m}]}. \tag{9.15}$$

The proof of this result is parallel to that of proposition 9.2 and we shall omit details. Indeed, this kind of estimates has also been studied for the solutions of $Q(t, d/dt)u = 0$ by Turrittin himself, which explains our terminology. It follows that Turrittin's equation admits a fundamental solution in second hyperfunctions for any m and k. The case when $a = i$, $m = 1$, and k is odd is of special interest, since we then obtain the Mizohata operator in the classically unsolvable case.

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