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NON LINEAR OPTIC AND SUPERCRITICAL WAVE EQUATION

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Dedicated to the memory of Pascal Laubin

1. Introduction and results

In this paper, we are interested in the study of the Cauchy problem for the non-linear wave equation in $\mathbb{R}^3$

$$(1.1)\qquad \begin{cases} (\partial_t^2 - \Delta_x)u + u^p = 0 \quad u = u(t,x) \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3 \\ u_{|t=0} = u_0(x) \in H^1 \cap L^{p+1} \ ; \ \partial_t u_{|t=0} = u_1(x) \in L^2 \end{cases}$$

Here, p is an odd integer, and the function u is assumed to take real values.

The formally conserved energy for (1.1) is

$$(1.2)\qquad E(u) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{u^{p+1}}{p+1} \right) dx$$

The Sobolev imbedding in $\mathbb{R}^3$, $H^1 \hookrightarrow L^6$, leads to the natural classification in terms of the different values of p

$p = 1$ linear
$p = 3$ subcritical
$p = 5$ critical
$p \geq 7$ super critical

Existence and uniqueness of strong solutions for (1.1) is well known in the subcritical case $p \leq 3$. In the critical case $p = 5$, the Cauchy problem for (1.1) has been solved by Grillakis [G] and Shatah-Struwe [S.S]. We recall here the known global result on strong solutions (see Shatah-Struwe [S.S] and Bahouri-Shatah [B.S]

Theorem 1. *($p = 5$) For any $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ there exist in the space*

$$B = \left\{ \partial_t u \in L^\infty(\mathbb{R}, L^2), \nabla_x u \in L^\infty(\mathbb{R}, L^2), u \in L^5(\mathbb{R}, L^{10}) \right\}$$

a unique solution to the Cauchy problem

$$\Box u + u^5 = 0 \ , \ u_{|t=0} = u_0 \ , \ \partial_t u_{|t=0} = u_1$$