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The behavior of solutions of some semilinear wave equations in one space dimension near their blow-up curve

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Dedicated to the memory of Pascal Laubin

1 Introduction and statement of the results

In this paper we shall consider the Cauchy problem

(1.1) $$\Box u = F(u, u') \text{ if } x \in \mathbb{R},\ t > 0,$$

(1.2) $$(\partial_t^j u)(x, 0) = \psi_j(x),\ j = 0, 1, \text{ if } x \in \mathbb{R},$$

where $\Box = \partial_t^2 - \partial_x^2$ is the d'Alembertian, $F \in C^1(\mathbb{R} \times \mathbb{R}^2)$, $u' = (\partial_x u, \partial_t u)$, and $\psi_j \in C^{2-j}(\mathbb{R})$, $j = 0, 1$. We shall put $\mathbb{R}^+ = \{s \in \mathbb{R},\ s > 0\}, \overline{\mathbb{R}^+} = \{s \in \mathbb{R},\ s \geq 0\}$. It is well known and easy to verify that one can find an open neighborhood V of $\mathbb{R} \times \{0\}$ in $\mathbb{R} \times \overline{\mathbb{R}^+}$ such that (1.1), (1.2) has a (unique) $C^2(V)$ solution. To be more precise, if $x \in \mathbb{R}$ and $t > 0$, put $K^-(x,t) = \{(y, s) \in \mathbb{R} \times \overline{\mathbb{R}^+},\ s < t,\ |x - y| < t - s\}$. If $\mathcal{U}$ is an open subset of $\mathbb{R} \times \overline{\mathbb{R}^+}$, one says that $\mathcal{U}$ is an influence domain if for any $(x, t) \in \mathcal{U}$, one has $\overline{K^-(x,t)} \subset \mathcal{U}$. Let Ω be the union of all influence domains containing $\mathbb{R} \times \{0\}$ in which (1.1), (1.2) has a (unique) C^2 solution. Then Ω is the maximal influence domain with that property. One can check that, for all $x \in \mathbb{R}$, $\{t > 0,\ \{x\} \times [0, t] \subset \Omega\} \neq \emptyset$. Put $\varphi(x) = \sup\{t > 0,\ \{x\} \times [0, t]\} \subset \Omega$. Then either $\varphi \equiv +\infty$ or φ is everywhere finite and $|\varphi(x) - \varphi(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. In [2], [3], under suitable assumptions on the initial data, a study was made of the case that $F(z, (p, q))$ is independent of (p, q), behaves like z^r, $r > 1$, as $z \to +\infty$, and is bounded below as $z \to -\infty$. When

φ is everywhere finite, it was shown that $u(x,t) \to +\infty$ as $t < \varphi(x)$ and $(x,t) \to (x_0, \varphi(x_0))$ for some $x_0 \in \mathbb{R}$. In this case, it was also proved in [2], [3] that $\varphi \in C^1(\mathbb{R})$. In the case of mixed problems and exponential nonlinearities, a study of such properties was made in [5], [6]. When F depends on (p,q) (and z), an example from [4] shows that φ need not be C^1. See also e.g. [1] and the references in [1], [2], [3], [4], [5], [6] for many results on the blow-up of solutions of nonlinear hyperbolic equations.

In this paper we shall consider two classes of functions $F(z,(p,q))$. Assuming φ to be everywhere finite, we shall obtain information on the behavior of $u(x,t)$, $u'(x,t)$, as $t < \varphi(x)$ and $(x,t) \to (x_0, \varphi(x_0))$ for some $x_0 \in \mathbb{R}$. We shall first prove the following result.

Theorem 1.1 *Assume that* $0 \leq F(z,(p,q)) \leq G(z)(1+|p|+|q|)$ *for all* $(z,(p,q)) \in \mathbb{R} \times \mathbb{R}^2$, *where* G *is non decreasing and* $e^{-\lambda G} \in L^1(\mathbb{R}^+)$ *for some* $\lambda > 0$. *Assume that* φ *is everywhere finite. Then* $u(x,t) \to +\infty$ *as* $t < \varphi(x)$ *and* $(x,t) \to (x_0, \varphi(x_0))$ *for some* $x_0 \in \mathbb{R}$.

We shall also consider the case that $F(z,p,q) = f(q-p)$ where f satisfies the following assumptions : $f \in C^1(\mathbb{R})$, $f \geq 0$, there exists $M \in \mathbb{R}$ such that $s \mapsto f(s)$ is non decreasing for $s \geq M$, and there exists $a \in \mathbb{R}$ such that $\displaystyle\int_{a+\epsilon}^{+\infty} \frac{ds}{f(s)} < +\infty$ for all $\epsilon > 0$, but $\displaystyle\int_{a}^{+\infty} \frac{ds}{f(s)} = +\infty$. We shall put $v_0 = \psi_1 - \psi_0'$ and assume for simplicity that $\{x \in \mathbb{R},\, v_0(x) > a\} =]\alpha, \beta[$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Then, of course $v_0(\alpha) = v_0(\beta) = a$. Put $\Sigma = \{(x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}, t = \displaystyle\int_{v_0(x-t)}^{\infty} \frac{ds}{f(s)}\}$. Then $\Sigma \neq \emptyset$ and it is easily checked that Σ is a C^1 curve. We shall assume for simplicity that

(1.3) there exists exactly one point $(x^*, t^*) \in \Sigma$ such that the tangent to Σ at (x^*, t^*) is parallel to $(-1,1)$; furtheremore $2v_0'(s) < f(v_0(s))$ if $x^* - t^* < s < \beta$.

Put $\mathcal{T} = \{(x^* - \mu, t^* + \mu), \mu > 0\}$, $\mathcal{E} = \Sigma \cup \mathcal{T}$. It is easy to check that , for all $x \in \mathbb{R}$, one can find $t > 0$ such that $(x,t) \in \mathcal{E}$. Put $\theta(x) = \inf\{t > 0, (x,t) \in \mathcal{E}\}$. One readily verifies that $|\theta(x) - \theta(y)| \leq |x-y|$, so $\{(x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}$, $t < \theta(x)\}$ is an influence domain.

Theorem 1.2 $|u'(x,t)| \to +\infty$ *as* $t < \theta(x)$ *and* $(x,t) \to (x_0, \theta(x_0))$ *for some* $x_0 \in \mathbb{R}$.

It follows in particular from Theorem 1.2 that $\theta = \varphi$, where φ is as in the beginning of this section. $u(x,t)$ itself may behave in several ways as $t < \theta(x)$ and $(x,t) \to (x_0, \theta(x_0))$, $x_0 \in \mathbb{R}$, as the next result shows.

Theorem 1.3 *Let f be as above.*

(1) Assume that for some $C > 0$ and some $b \leq 3$, $f(s) \leq Cs^b$ for large s. Then one can find ψ_0, ψ_1 such that (1.3) holds and such that $u(x,t) \to +\infty$ as $t < \theta(x)$ and $(x,t) \to (x^*, t^*)$.

(2) Assume that for some $C > 0$ and some $b > 3$, $f(s) \geq Cs^b$ for large s. Then one can find ψ_0, ψ_1 such that (1.3) holds and such that $u(x,t)$ has a finite limit as $t < \theta(x)$, $x + t \leq x^* + t^*$ and $(x,t) \to (x^*, t^*)$.

Our paper is organized as follows. Theorem 1.1 is proved in section 2; Theorem 1.2 and Theorem 1.3 are proved in section 3.

2 Proof of Theorem 1.1

We shall make use of the following well known result, of which we give a proof for the sake of completeness.

Lemma 2.1 *Assume that for some $C > 0$, one has $|F(z,(p,q))| \leq C(1 + |p| + |q|)$ for all $(z,(p,q)) \in \mathbb{R} \times \mathbb{R}^2$. Then $\Omega = \mathbb{R} \times \overline{\mathbb{R}^+}$.*

To prove Lemma 2.1, it is enough to show that for any $(X,T) \in \mathbb{R} \times \overline{\mathbb{R}^+}$, there exists $C > 0$ such that the following holds :

(2.1)

if $0 < T_1 \leq T$ and $u \in C^2(\{(x,t) \in K^-(X,T),\ t < T_1\})$ is a solution of (1.1), (1.2) when $(x,t) \in K^-(X,T), t < T_1$, then $|\partial^\alpha u| \leq C$ if $|\alpha| \leq 2$.

To verify (2.1), put $P(x,t) = \frac{1}{2}(\psi_0(x-t) + \psi_0(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi_1(y)\,dy$ and write (1.1), (1.2) in the form

$$u(x,t) = P(x,t) + \frac{1}{2} \iint_{K^-(x,t)} (F(u,u'))(y,s)\,dy\,ds \tag{2.2}$$

For $0 \leq t < T_1$, put $M(t) = \sup_{(x,t) \in K^-(X,T)} |u'(x,t)|$. By differentiation of (2.2), it follows that, for some $C_1 > 0$,

$$M(t) \leq C_1 + C_1 \int_0^t M(s)\, ds, \tag{2.3}$$

so $M(t) \leq C_1 e^{C_1 T}$ if $0 \leq t < T_1$ by the Gronwall inequality. Differentiating (2.2) a second time easily yields (2.1).

To prove Theorem 1.1, let $\chi : \mathbb{R} \to [0,1]$ be a C^1 function with $\chi(s) = 1$ if $s \leq 1$ and $\chi(s) = 0$ if $s \geq 2$. For $k \in \mathbb{N} \setminus \{0\}$, consider the Cauchy problem

$$\Box u_k = F(\chi\left(\frac{u_k}{k}\right) u_k, u_k') \text{ if } x \in \mathbb{R},\ t > 0, \tag{2.4}$$

$$(\partial_t^j u_k)(x,0) = \psi_j(x), \quad j = 0,1, \qquad \text{if } x \in \mathbb{R}. \tag{2.5}$$

By Lemma 2.1, (2.4), (2.5) has a unique solution $u_k \in C^2(\mathbb{R} \times \overline{\mathbb{R}^+})$. We are going to adapt some ideas of [3] to the present situation. Let $((X_j, T_j))$ be a dense sequence in $\mathbb{R} \times \overline{\mathbb{R}^+}$. Since $F \geq 0$, it follows (cf. [3]) that $(u_k(x,t))$ is bounded below at each (x,t). Passing to a subsequence of (u_k) if necessary, one can assume that for each (X_j, T_j), either (i) $u_k(X_j, T_j)$ has a finite limit as $k \to +\infty$, or (ii) $u_k(X_j, T_j) \nearrow +\infty$ as $k \to +\infty$. In case (i), we shall say that $(X_j, T_j) \in \mathcal{E}_1$, whereas in case (ii) we shall say that $(X_j, T_j) \in \mathcal{E}_2$. If $\mathcal{E}_2 = \emptyset$, one checks easily with the help of a formula of type (2.2) and of the Ascoli theorem that (u_k) has a subsequence $(u_{k'})$ such that $(\partial^\alpha u_{k'})$ converges uniformly on every compact subset of $\mathbb{R} \times \overline{\mathbb{R}^+}$ if $|\alpha| \leq 2$; so in that case (1.1), (1.2) has a global solution. If $(x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}$, put $K^+(x,t) = \{(y,s) \in \mathbb{R} \times \mathbb{R}^+, s > t, |x-y| < s - t\}$. If $x \in \mathbb{R}$, define $A_x = \{t > 0$, there exists $(X_j, T_j) \in \mathcal{E}_1$ such that $(x,t) \in \overline{K^-(X_j, T_j)}\}$, and if $\mathcal{E}_2 \neq \emptyset$, define $B_x = \{t > 0$, there exists $(X_j, T_j) \in \mathcal{E}_2$ such that $(x,t) \in \overline{K^+(X_j, T_j)}\}$. Assuming that $\mathcal{E}_2 \neq \emptyset$, one checks easily that $\sup A_x = \inf B_x$. This number will be denoted by $\Phi(x)$. The following properties are easily checked :

1. $|\Phi(x) - \Phi(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
2. One can find a subsequence $(u_{k'})$ of (u_k) and a function $\tilde{u}$ in the space $C^2\left(\{(x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}, t < \Phi(x)\}\right)$ such that, for all $(x_0, t_0) \in \mathbb{R} \times \overline{\mathbb{R}^+}$ with $t_0 < \Phi(x_0)$, $u_{k'} \to \tilde{u}$ in $C^2(\overline{K^-(x_0, t_0)})$ as $k' \to +\infty$, whereas $u_{k'}(x,t) \to +\infty$ as $k' \to +\infty$ for any (x,t) with $t > \Phi(x)$. To prove Theorem 1.1, it is enough to show that $\tilde{u}(x,t) \to +\infty$ if $t < \Phi(x)$ and $(x,t) \to (x_0, \Phi(x_0))$ for some $x_0 \in \mathbb{R}$. This will also show that $\Phi = \varphi$.

Let P be as in (2.2) and put $F_k = F\left(\chi\left(\frac{u_k}{k}\right) u_k, u_k'\right)$. By the formula already used in (2.2), we have

$$u_k(x,t) = P(x,t) + \frac{1}{2} \iint_{K^-(x,t)} F_k(y,s)\, dy\, ds \tag{2.6}$$

Fix $(X,T) \in \mathbb{R} \times \overline{\mathbb{R}^+}$ with $T > \Phi(X)$, and define, if $0 \leq T' \leq T$,

$$M_k(T',t) = \sup_{(x,t) \in \overline{K^-(X,T')}} |u_k'(x,t)|.$$

Now, for some $C > 0$ independent of $T' \in [0,T]$, $u_k(y,s) \leq u_k(X,T') + C$ when $k \in \mathbb{N} \setminus \{0\}$ and $(y,s) \in \overline{K^-(X,T')}$. Hence, differentiating (2.6) (as in the proof of Lemma 2.1) and using the fact that G is non decreasing, we obtain, for some $C_1 > 0$ independent of k, t, T' when $t \leq T' \leq T$:

$$M_k(T',t) \leq C_1 + G(u_k(X,T') + C) \int_0^t (1 + M_k(T',s))\, ds. \tag{2.7}$$

Adding 1 to both members of (2.7) and applying the Gronwall inequality, we deduce that

$$M_k(T',T') \leq (C_1 + 1) e^{T'G(u_k(X,T')+C)}.$$

This implies in particular that

$$\partial_t u_k(X,t) \leq (C_1 + 1) e^{TG(u_k(X,t)+C)} \tag{2.8}$$

if $0 \leq t \leq T$. We may assume that T is so large that $e^{-TG} \in L^1(\mathbb{R}^+)$. Put $g(s) = -\int_s^\infty e^{-TG(\sigma)} d\sigma$. (2.8) means that

$$\frac{d}{dt}\left(g(u_k(X,t) + C)\right) \leq C_1 + 1 \tag{2.9}$$

if $0 \leq t \leq T$. Take t_1, t_2 with $0 \leq t_1 < \Phi(X) < t_2 \leq T$. Integrating (2.9) over $[t_1, t_2]$, we obtain that

$$g(u_k(X,t_2) + C) - g(u_k(X,t_1) + C) \leq (C_1 + 1)(t_2 - t_1).$$

But $e^{-TG} \in L^1(\mathbb{R}^+)$. So if $k \to +\infty$, it follows that

$$-g(\tilde{u}(X,t_1) + C) \leq (C_1 + 1)(t_2 - t_1).$$

When $t_2 \xrightarrow{>} \Phi(X)$, we obtain that

$$-g(\tilde{u}(X,t_1)+C) \leq (C_1+1)(\Phi(X)-t_1). \tag{2.10}$$

Now the proof leading to (2.10) also shows that (2.10) still holds with uniform constants $C, C_1 > 0$ when X belongs to a compact neighborhood of $x_0 \in \mathbb{R}$ and $0 \leq t_1 < \Phi(X)$. Hence $g(\tilde{u}(X,t_1)+C) \to 0$ as $t_1 < \Phi(X)$ and $(X,t_1) \to (x_0, \Phi(x_0))$, which completes the proof of Theorem 1.1.

3 Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2.

Assume first that $(x_0, \theta(x_0)) \in \Sigma$. Put $(\partial_t - \partial_x)u = v$. Then of course $(\partial_t + \partial_x)v = f(v)$ and $v|_{t=0} = v_0$. Solving this Cauchy problem for v, we can easily check that $v(x,t) \to +\infty$ as $t < \theta(x)$ and $(x,t) \to (x_0, \theta(x_0))$. We are going to show that the following holds :

$$(\partial_t + \partial_x)u(x,t) \to +\infty \text{ as } x+t < x^* + t^* \text{ and } (x,t) \to (x^*, t^*). \tag{3.1}$$

Since $\Box u \geq 0$, it will follow from (3.1) that $(\partial_t + \partial_x)u(x,t) \to +\infty$ as $x+t < x^* + t^*$ and $(x,t) \to (x_0, \theta(x_0))$ if $(x_0, \theta(x_0)) \in \mathcal{E} \setminus \Sigma$. This will complete the proof of Theorem 1.2.

If $\sigma > a$, put $h(\sigma) = \displaystyle\int_\sigma^\infty \frac{ds}{f(s)}$. If $\alpha < x - t < \beta$ and $0 \leq t < \theta(x)$, we certainly have $v(x,t) > a$ and $h(v(x,t)) = h(v_0(x-t)) - t$. Hence $v(x,t) = H(h(v_0(x-t))-t)$ if $H :]0,+\infty[\to]a,+\infty[$ is the inverse function of $h :]a,+\infty[\to]0,+\infty[$. Put $\xi = \dfrac{t+x}{2}$, $\eta = \dfrac{t-x}{2}$, $\xi^* = \dfrac{t^*+x^*}{2}$, $\eta^* = \dfrac{t^*-x^*}{2}$, $U(\xi,\eta) = u(x,t)$, $\gamma(s) = h(v_0(-2s)) - s$. By the definition of v, we have

$$U(\xi,\eta) = U(\xi,-\xi) + \int_{-\xi}^{\eta} v(\xi - s, \xi + s)\, ds, \tag{3.2}$$

$$(\partial_\xi U)(\xi,\eta) = Q(\xi) + \int_{-\xi}^{\eta} (f(v))(\xi - s, \xi + s)\, ds, \tag{3.3}$$

where $Q \in C^1(\mathbb{R})$. Notice that

$$v(\xi - s, \xi + s) = H(\gamma(s) - \xi) \tag{3.4}$$

in (3.2), (3.3). Since $\gamma(\eta^*) = \xi^*$, we easily obtain that

$$\gamma(s) - \xi \leq \xi^* - \xi + c|s - \eta^*| \tag{3.5}$$

for some $c > 0$, if s is close to η^*. In (3.3) assume that $\xi < \xi^*$, that s is close to η and (ξ, η) close to (ξ^*, η^*). It then follows from (3.4), (3.5) that

$$(f(v))(\xi - s, \xi + s) \geq (f \circ H)(\xi^* - \xi + c|s - \eta^*|). \tag{3.6}$$

Assume now that $\xi_k < \xi^*$ if $k \in \mathbb{N} \setminus \{0\}$ and that $(\xi_k, \eta_k) \to (\xi^*, \eta^*)$ as $k \to +\infty$. We are going to show that $(\partial_\xi U)(\xi_k, \eta_k) \to +\infty$ as $k \to +\infty$, which will prove (3.1) and hence Theorem 1.2. Let $\delta > 0$ be small and put $f_k(s) = \mathbf{1}_{[\eta_k - \delta, \eta_k]}(s) f(H(\xi - \xi_k + c|s - \eta^*|))$ if $k \in \mathbb{N} \setminus \{0\}$ and $s \in \mathbb{R}$, where $\mathbf{1}_A$ denotes the characteristic function of the set A. Since $f \circ H = -H'$, it easily follows with the help of (3.6) that $\int_{\mathbb{R}} (\liminf_{k \to +\infty} f_k(s))\, ds = +\infty$. Hence, by the Fatou lemma, we obtain that $\liminf_{k \to +\infty} \int_{\mathbb{R}} f_k(s)\, ds = +\infty$. (3.3) then shows that $(\partial_\xi U)(\xi_k, \eta_k) \to +\infty$ as $k \to +\infty$.

Proof of Theorem 1.3

(1) Take $\psi_j \in C^{3-j}(\mathbb{R})$, so that $v_0 \in C^2(\mathbb{R})$. Since $\gamma'(\eta^*) = 0$, we find that $H(\gamma(s) - \xi) \geq$ $C_1 |s - \eta^*|^{-\frac{2}{b-1}}$ for some $C_1 > 0$ if (ξ, s) is close to (ξ^*, η^*). Theorem 1.3(1) follows easily with the help of (3.2), (3.4) if we make use of the Fatou lemma.

(2) It is not hard to check that one can choose $v_0 \in C^2(\mathbb{R})$ such that $\gamma''(\eta^*) > 0$ (and of course $\gamma(\eta^*) - \xi^* = \gamma'(\eta^*) = 0$). Then we find that $H(\gamma(s) - \xi) \leq C_2 |s - \eta^*|^{-\frac{2}{b-1}}$ for some $C_2 > 0$ if (ξ, s) is close to (ξ^*, η^*) and $\xi \leq \xi^*$. Theorem 1.3(2) follows easily with the help of (3.2), (3.4).

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