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Optimization by n-homogeneous polynomial perturbations *

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We deal with infinite dimensional optimization in Banach spaces, finding an existence result for maximum (or minimum) points for a certain type of functions.

A remarkable result in this direction is the Stegall variational principle [10]: if C is a nonempty, closed, bounded and convex subset of a Banach space, C has the Radon-Nikodym property and f is an upper bounded upper semicontinuous real-valued function on C, then there exists an arbitrarily small linear continuous perturbation φ such that $f+\varphi$ attains its strong maximum on C. Our aim in this note is to obtain a Stegall's type result showing that we have an arbitrarily small continuous n-homogeneous polynomial perturbation (n-odd natural number) with the same property.

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There are many other variational principles, for example, Ekeland's variational principle [6], which has the same hypotheses on the function to be maximized but without assuming the Radon-Nikodym property, and gives a small perturbation which is only Lipschitz. A general question in the optimization theory is to find good perturbations (assuring existence of a point of minimum of the perturbed function) in some class of perturbations. In many cases the set ot good perturbations is dense (even G_δ) in the set of all perturbations. For further information about variational principles and perturbed optimization see for example : [3-10].

Our result in this note is related, as well, to the paper of R.Aron, C.Finet and E.Werner [2], where an extension of the Bishop-Phelps theorem is proved for n-linear continuous forms in some spaces. Their arguments shows also the denseness of norm-attaining n-homogeneous polynomials in a Banach space with the Radon-Nikodym property. M.D.Acosta, F.J.Aguirre and R.Paya [1] constructed examples, showing that the results in [2] are not valid in arbitrary Banach spaces.

Let $(X, \|.\|)$ be a Banach space. We shall recall the following definitions.

Definition 1 *(a) Suppose that $C \subset X$ is a nonempty set and that f is an upper bounded real-valued function on C. For each $\alpha \geq 0$ define the slice of C by*

$$S(f, \alpha) = \{x \in C : f(x) \geq \sup f(C) - \alpha\}$$

(b) A nonempty subset A of X is said to be dentable provided it admits slices of arbitrarily small diameter, that is, for every $\varepsilon > 0$ there exists $x^* \in E^*$ and $\alpha > 0$ such that $diamS(x^*, A, \alpha) < \varepsilon$.

Definition 2 *A subset A of X is said to have the Radon-Nikodym property if every nonempty bounded subset of A is dentable.*

We say that f attains its strong maximum at x over C, if $f(x) = \sup f(C)$ and $\|x - x_n\| \to 0$ whenever $f(x_n) \to f(x)$.

Let us denote by $F_n(X)$ the space of all continuous and symmetric n-linear forms on $X \times ... \times X$ into $\mathbf{R}$ endowed with the norm

$$\|A\| = \sup\{|A(x_1, ..., x_n)| \ : \|x_i\| \leq 1, \quad i = 1, ..., n\}.$$

If $A \in F_n(x)$, then consider the function $\varphi_A : X \to \mathbf{R}$ defined by $\varphi_A(x) = A(x, ..., x)$; φ_A is called n-homogeneous polynomial on X. We denote by $P_n(X)$ the Banach space of all continuous n-homogeneous polynomials on X, endowed with the norm:

$$\|\varphi_A\| = \sup\{|\varphi_A(x)|, \|x\| \leq 1\}$$

Theorem 3 *Suppose that $C \subset X$ is a nonempty, bounded, closed and convex set with the Radon-Nikodym property. Let f be an upper bounded upper semicontinuous real-valued function on C. Then for every odd integer n there exists a dense G_δ subset Q_n of $P_n(X)$ such that for every $\varphi \in Q_n$, $f + \varphi$ attains its strong maximum on C.*

Our proof follows the ideas of the proof of Stegall's variational principle [10], given in the book of Phelps [8]. We need the following lemmas.

Lemma 4 ([8]). *Suppose that $\{A_n\}_{n=1}^{\infty}$ is a sequence of nonempty subsets of X with the following property: there exist constants $\varepsilon > 0$ and $\lambda > 0$ such that for all $x \in coA_n$ and $y \in X$*

$$dist[x, co(A_{n+1} \setminus B(y; \varepsilon))] \leq \lambda/2^n$$

Then the set

$$A := \cap_{n=1}^{\infty} \overline{\cup_{j \geq n} coA_j}$$

is nonempty and not dentable.

The proof of the following lemma is straightforward and is omitted.

Lemma 5 ([8]). *Suppose that the real-valued function f is bounded above on the nonempty subset C of X. Then for every $\alpha > 0$, there exists $\beta > 0$ such that $S(f + \varphi, \beta) \subset S(f, \alpha)$, whenever $\varphi \in P_n(X)$ and $\|\varphi\| < \beta$.*

Lemma 6 . *Let C be a nonempty, bounded, closed and convex subset of X with the Radon-Nikodym property, f be an upper semicontinuous, real-valued, bounded above function on C and n be an odd integer number. Then for any $\varepsilon > 0$ there exist $\varphi \in P_n(X), \|\varphi\| < \varepsilon$, and $\alpha > 0$ such that diam $S(f + \varphi, \alpha) \leq 2\varepsilon$.*

Proof. Proceeding by contradiction, suppose that there exists $\varepsilon > 0$ such that for every $\varphi \in P_n(X), \|\varphi\| < \varepsilon$ and each $\alpha > 0$, we have

$$\text{diam } S(f + \varphi, \alpha) > 2\varepsilon$$

For each m let

$$A_m = \cup\{S(f + \varphi, \frac{1}{4^{mn}}) : \varphi \in P_n(X), \|\varphi\| \leq \varepsilon - \frac{1}{2^m}\}.$$

The set A_m is nonempty. Take $\lambda = 5/2$. We will show that the sequence $\{A_m\}_{m=1}^{\infty}$ satisfies the hypothesis of Lemma 4, which will give us a contradiction, since C has the Radon-Nikodym property. We want to show that, for any natural m and $y \in X$,

$$coA_m \subset co(A_{m+1} \setminus B(y; \varepsilon)) + \frac{\lambda}{2^m} B(0; 1). \tag{1}$$

Let m be fixed natural number. Since the set of the right side is convex, it suffices to prove that it contains A_m. Suppose that $x \in A_m$ but for some $y \in X$, it is not in the right hand side of (1).

By the separation theorem, there exists $y^* \in X^*, \|y^*\| = 1$ such that

$$\langle y^*, x\rangle > \sup\{\langle y^*, a\rangle : a \in A_{m+1} \setminus B(y; \varepsilon)\} + \frac{\lambda}{2^m}$$

Then

$$\langle y^*, x\rangle^n > (\langle y^*, a\rangle + \frac{\lambda}{2^m})^n \quad \forall a \in A_{m+1} \setminus B(y; \varepsilon) \tag{2}$$

Now suppose that $x \in S(f + \varphi, \frac{1}{4^{nm}})$ with $\varphi \in P_n(X)$ and $\|\varphi\| \leq \varepsilon - \frac{1}{2^m}$. Consider

$$P(z) = \varphi(z) + r\langle y^*, z\rangle^n$$

with $r = \frac{1}{2^{m+1}}$. As $\varphi \in P_n(X)$, let H in $F_n(X)$ be the corresponding symmetric form. Then

$$P(z) = L(z, ..., z),$$

where

$$L(x_1, ..., x_n) = H(x_1, ..., x_n) + r\langle y^*, x_1\rangle ... \langle y^*, x_n\rangle.$$

Therefore $L \in F_n(X), P \in P_n(X)$ and

$$\|P\| \le \|\varphi\| + r \le \varepsilon - \frac{1}{2^{m+1}}.$$

Then $S(f+P, \frac{1}{4^{(m+1)n}})$ is contained in A_{m+1}. But as $P \in P_n(X)$ and $\|P\| < \varepsilon$, we have

$$\text{diam } S(f+P, \frac{1}{2^{(m+1)n}}) > 2\varepsilon.$$

Then there exists z in $S(f+P, \frac{1}{4^{(m+1)n}}) \setminus B(y; \varepsilon)$ and

$$\begin{aligned}
(f+P)(z) &\ge \sup(f+P)(C) - \tfrac{1}{4^{(m+1)n}} \\
&\ge (f+P)(x) - \tfrac{1}{4^{(m+1)n}} \\
&= (f+\varphi)(x) + r\langle y^*, x\rangle^n - \tfrac{1}{4^{(m+1)n}} \qquad \text{(by definition of } P) \\
&\ge \sup(f+\varphi)(C) - \tfrac{1}{4^{mn}} + r\langle y^*, x\rangle^n - \tfrac{1}{4^{(m+1)n}} \\
&\qquad (\text{as} \quad x \in S(f+\varphi, \tfrac{1}{4^{mn}})) \\
&> (f+\varphi)(z) - \tfrac{1}{4^{mn}}(1 + \tfrac{1}{4^n}) + r(\langle y^*, z\rangle + \tfrac{\lambda}{2^m})^n \quad \text{(by (2))} \\
&= (f+P)(z) - r\langle y^*, z\rangle^n - \tfrac{1}{4^{mn}}(1 + \tfrac{1}{4^n}) + r(\langle y^*, z\rangle + \tfrac{\lambda}{2^m})^n.
\end{aligned}$$

It follows that

$$2^{m(1-n)+1}(1 + \frac{1}{4^n}) > 2^{mn}(((\langle y^*, z\rangle + \frac{\lambda}{2^m})^n - \langle y^*, z\rangle^n). \tag{3}$$

Case 1. $n = 1$.
Then by (3) we obtain $\frac{5}{2} > \lambda$, a contradiction.
Case 2. $n \ge 3$ and $\langle y^*, z\rangle \ne 0$.
Consider the real function

$$\xi(t) = (at + \lambda)^n - (at)^n,$$

where $a = \langle y^*, z\rangle$ (recall that n is an odd natural number). It is easy to see that $\lim_{t\to\pm\infty} = +\infty, \xi$ attains its global minimum over $\mathbf{R}$ at the point $t_0 = -\frac{\lambda}{2a}$ and $\xi(t_0) = 2(\frac{\lambda}{2})^n$. Then the left side of (3) is less than $2(\frac{\lambda}{2})^n$, a contradiction with (3).

Case 3. $n \geq 3$ and $\langle y^*, z\rangle = 0$.

By (3) we obtain a contradiction. The lemma is proved.

Proof of Theorem 3. Fix n and for every natural m define

$$Q_n^m = \{\varphi \in P_n(X): \text{ diam } S(f+\varphi, \alpha_m) < \frac{1}{m} \quad \text{for some } \alpha_m > 0\}.$$

By Lemma 6, for each m, Q_n^m is dense in $P_n(X)$ and it is open by Lemma 5. By the Baire cathegory theorem, the set

$$Q_n = \cap_m Q_n^m$$

is a dense G_δ subset of $P_n(X)$.

Since f is upper semicontinuous, the set $S(f+\varphi, \alpha_m)$ is a closed set. Let x_0 be the unique common point of the sets $S(f+\varphi,\alpha_m), m \geq 1$ and let $\varphi \in Q_n, \{x_n\} \subset C$, $(f+\varphi)(x_n) \to \sup_{x\in C}(f+\varphi)(x)$. Then for every $m \geq 1$ there exists ν such that $x_n \in S(f+\varphi, \alpha_m)$ for every $n > \nu$. This means $\|x_n - x_0\| < \frac{1}{m}$, i.e. $x_n \to x_0$ and the theorem is proved.

It is easy to see that Theorem 3 is not valid for even n: take, for example, $X = \mathbf{R}$, $C = [0,1]$, $n = 2$ and $f(x) = x^2$.

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