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Differentiability of functions with values in some real associative algebras: approaches to an old problem

by

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Abstract

During the last one hundred and fifty years several mathematicians and physicists considered the question of generalizing real or complex differentiability and derivability to the case of algebra-valued functions and their relation to higher dimensional function theories. In this paper we give an overview of some attempts and approaches.

1 General remarks and historical review

1.1 Origins of the problem

The classical definitions of differentiability and the differential as the linear part of the increment can be extended in several ways. The extension to functions defined on a real or complex Banach space is one of the most general ones. It leads to the concepts of differentiability in the sense of Gâteaux or Fréchet.

The concrete determination of the differential, i.e. the approximation of the function considered in a neighbourhood of some point by a linear mapping, is an important operation in differential calculus in finite dimensional real or complex vector spaces.
As an example we refer to the well-known fact that if $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, then the derivative of a real differentiable function $f : X \to Y$ is defined by the Jacobian matrix which is a continuous $\mathbb{R}$-linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$.
This type of derivative is a directional derivative: it is related to the fact that division by an element of the scalar field is possible.

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But in the case of $\mathbb{R}^1$ and $\mathbb{C}$ this scalar field can coincide with the elements of the vector space itself and then the derivative - a (1×1)-matrix - reduces to a single function. In these cases differentiability (or derivability, considered as the equivalent property that the derivative is a well-defined function) can simply be defined by requiring the (local) existence of the limit of the difference quotient.

Moreover, the case of a complex-valued function of a complex variable yields a qualitatively new effect compared with the $\mathbb{R}^2$ setting: if the limit exists, then this implies immediately the independence of the direction in which the limit is obtained. As a consequence, such functions - belonging to the well-known class of holomorphic functions - satisfy the Cauchy-Riemann system. Notice hereby that $\mathbb{C}$ enriches the character of $\mathbb{R}^2$: it is a real division algebra and a field.

As the famous theorems of Frobenius, respectively Hurwitz state (see e.g. [54]) the algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ are the only real associative division algebras, while the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only normed real algebras. Hereby, $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ stand for the algebras of real numbers, complex numbers and real quaternions, while $\mathbb{O}$ denotes the algebra of Cayley numbers (real octonions).

Since $\mathbb{H}$ is a skew field, it thus seems natural to ask whether differentiability of a function $f : \mathbb{H} \to \mathbb{H}$ can be defined in a similar way as in the cases $\mathbb{R}$ or $\mathbb{C}$. This question was posed already at the end of the 19th century. However, it took almost a century to find the right answer or, more exactly, the right answers.

Due to the historical development we do not only consider quaternion-valued functions of a quaternionic variable but also functions with values in a Clifford algebra. Indeed, $\mathbb{C}$ and $\mathbb{H}$ are special cases of real Clifford algebras, but there is more: although the latter algebras are in general not division algebras, they allow Cauchy-type theorems and thus lead in a natural way to a function theory in higher dimensional Euclidean spaces.

At the beginning of December 2000, Pascal Laubin asked the first author for information about the problem of differentiability of functions of a quaternionic variable. The circumstances we know have not allowed an extensive exchange of ideas and results on this subject. The authors of the present paper hope that the survey given may shed some light on this problem.

1.2 Attempts with unsatisfactory results

If one thinks of generalizing the concept of differentiability and derivability to higher dimensions, then classical real and complex calculus suggest to consider differential quotients.
The first step in this sense was made by R. W. Hamilton in 1860, who - as is well known - invented the quaternions in 1843.
In trying to introduce a three-dimensional analogue to the complex number system, he found out that his new numbers should have four real components and that the commutative law of multiplication should be abandoned.
The algebra $\mathbb{H}$ of real quaternions that he constructed has three imaginary units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ which satisfy the multiplication rules

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j},$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

A real quaternion z is then a number of the form

$$z = x_0\,1 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \quad x_j \in \mathbb{R},\ j = 0, \ldots, 3.$$

Defining the anti-involution $z \mapsto \overline{z}$ on $\mathbb{H}$ by

$$\overline{z} = x_0\,1 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k},$$

one obtains that

$$z\,\overline{z} = \overline{z}\,z = \|z\|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

whence any $z \in \mathbb{H}, z \neq 0$, has an inverse

$$z^{-1} = \frac{\overline{z}}{\|z\|^2}.$$

As already mentioned, the algebra $\mathbb{H}$ is not commutative. One has e.g. that $\mathbf{ij} = \mathbf{k}$ while $\mathbf{ji} = -\mathbf{k}$. For a function $f : \mathbb{H} \to \mathbb{H}$, R. W. Hamilton considered in [19] a limit of the form

$$\lim_{t \to 0} \frac{f(z + t\Delta z) - f(z)}{t}, \quad t \in \mathbb{R}, \quad z \in \mathbb{H} \tag{1}$$

which, however, is not the limit of a quotient generally defined by the two quaternion-valued increments of function and argument.

A quaternionic differential quotient is a limit of the form

$$\lim_{\Delta z \to 0} \Big(f(z + \Delta z) - f(z)\Big)\Big(\Delta z\Big)^{-1} \tag{2}$$

or

$$\lim_{\Delta z \to 0} \Big(\Delta z\Big)^{-1}\Big(f(z + \Delta z) - f(z)\Big). \tag{3}$$

Observe that these differential quotients can be defined since $\mathbb{H}$ is a skew field.

One of the first authors who dealt with such kind of differential quotients in general division algebras generalizing $\mathbb{R}$ and $\mathbb{C}$ was G. Scheffers in 1893 (see [46]). He considered the example of the function $f(z) = z^2$ and concluded from it that one can only find in commutative division algebras non-constant functions for which limits in the sense of (2) and (3) exist and are uniquely defined. For this reason G. Scheffers restricted himself to the case of commutative division algebras. However, his assertion is not correct, since the limit (2) exists for quaternionic functions $f(z) = az + b$ where $a, b \in \mathbb{H}$, and the limit (3) for quaternionic functions $f(z) = za + b$. If a is a real number, then both limits exist and equal a.

Nevertheless, his paper had an important influence on the development of quaternionic analysis. R. Fueter, one of the founders of quaternionic and Clifford analysis (cf. [10, 11, 13]), mentioned in [10] the paper of G. Scheffers in order to motivate a generalization of complex analyticity to the quaternionic case by means of a different approach, namely by the so-called Riemann approach which is based on considering quaternionic valued functions in the kernel of the quaternionic Cauchy-Riemann operator $\mathcal{D} := \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1}\mathbf{i} + \frac{\partial f}{\partial x_2}\mathbf{j} + \frac{\partial f}{\partial x_3}\mathbf{k}$.

And in fact, we will see that generalizing complex analysis to quaternions by using the approach (2) or (3) is not very fruitful.

As far as we know, the first authors who analyzed exhaustively for which quaternionic functions the expressions (2) or (3) exist were N. M. Krylov in 1947 and his student A.S. Melijhzon in 1948. In [37] one can find a detailed proof of the property that functions of the type $f(z) = az + b$ are the most general ones for which (2) exists. Analogously, functions of the form $f(z) = za + b$ are the only ones satisfying (3). In [23] the case (3) is already treated. Hence, a generalization of differentiability in the sense of (2) or (3) leads to a very restrictive class of functions.

Because of a lack of scientific communication, the results of N. M. Krylov and A.S. Melijhzon remained unknown for quite a long time. J. J. Buff proved e.g. in 1973 in [4] that functions of the type $f(z) = \alpha z + b$, where α is a real number, are the only ones for which both limits exist.

In the same year C. A. Deavours published a paper on quaternionic calculus (cf. [5]). He referred to the work of R. W. Hamilton and mentioned expressions of the form (2) and (3). He observed that $f(z) = z^2$ is not quaternionic differentiable in this sense, which motivated him to discuss immediately the Riemann approach.

In 1979 A. Sudbery also considered the problem of generalizing the concept of differentiability to quaternions. He gives in [51] an elegant proof of A. S. Melijhzon's and J. J. Buff's results by identifying $\mathbb{H}$ with $\mathbb{C}^2$ and using arguments from the theory of complex analysis in two variables.
The arguments of A. Sudbery justify in a very convincing way and this in accordance with R. Fueter, that the already mentioned Riemann approach may be considered as being the only successful way for generalizing classical complex function theory to the case of functions of a quaternionic variable. But it is surprising to see that neither A. Sudbery's paper from 1979 nor the first monograph in the field of Clifford analysis [2] from 1982 brought to an end the discussion about possible concepts of quaternionic differentiability. Almost ten years later (1991) the article [55] proposed some very formal constructions of partial derivatives with respect to a quaternionic variable, mainly for simplifying calculations, but without describing the class of functions with such properties. In a series of articles ([27],[28],[29]) one can find similar attempts made by S. Lugojan and this until the 90's. In these papers the author tried to avoid problems with the limit of the difference quotient by using an equivalent definition of the derivative as derivative in the sense of Fréchet of a function $f : \mathbb{H} \to \mathbb{H}$. But the class of differentiable functions in this sense contains again only linear functions. Moreover, no further applications or conclusions are given.

In the following sections we will see that a careful adaptation of the general methods of real and complex analysis together with the use of some specific properties of the non-commutative algebras considered lead to the characterization of two important classes of functions by means of suitable differentiability properties.

2 Quaternionic differentiability

In this section it will be shown how a subtle use of non-constant structural sets in $\mathbb{H}$ leads to a class of differentiable functions consisting of all Möbius transformations in $\mathbb{R}^4$ and constant functions.

2.1 Structural sets and differential forms of the first order

For the sake of simplicity, we put $\mathbf{i} = i_1, \mathbf{j} = i_2$ and $\mathbf{k} = i_3$ and $i_0 = 1$.
For $z \in \mathbb{H}$, $Re(z) = x_0$ is called the real part and $Pu(z) = x_1 i_1 + x_2 i_2 + x_3 i_3$ its pure imaginary part.
On $\mathbb{H}$ one can introduce a scalar product via $< a, b >:= Re(\bar{a}b)$. The induced norm on $\mathbb{H}$ coincides with the Euclidean norm in $\mathbb{R}^4$.

Let $\Omega \subset \mathbb{H}$ be an open set and let $f : \Omega \to \mathbb{H}$ be a quaternion valued function. Then with respect to the standard basis (i_0, i_1, i_2, i_3) of $\mathbb{H}$ we may write

$$f(z) = \sum_{j=0}^{3} i_j f_j(z),$$

where f_j are real valued. If f is real differentiable in the usual sense, then its differential is given by

$$df = \sum_{j=0}^{3} \frac{\partial f}{\partial x_j} dx_j, \tag{4}$$

The differential of the identity function $z \mapsto z$ is then

$$dz = \sum_{j=0}^{3} i_j dx_j. \tag{5}$$

Passing to a more general basis we introduce

Definition 1. *A set of four quaternions $[\Psi] := \{\Psi_0, \Psi_1, \Psi_2, \Psi_3\}$ is called a structural set, if $<\Psi_j, \Psi_k>= \delta_{jk}$ for $0 \le j, k \le 3$, where δ_{jk} stands for the Kronecker symbol.*

Structural sets have been introduced and widely used in hypercomplex function theory, e.g. in the works of M. Shapiro, N. Vasilevski and V.V. Kravchenko (cf. e.g. [24, 50]).
Usually, it is assumed that it is sufficient to consider structural sets of "constant" type, i.e. obtained by applying a rotation to the standard basis (see [14]). In what follows it will be shown that it may be useful to consider 'non-constant" structural sets as well.

2.2 Linear fractional functions

In [21, 22] the following notion of quaternionic differentiability was introduced:

Definition 2. *Let $\Omega \subset \mathbb{H}$ be an open set and let $z^* \in \Omega$ with $z^* = x_0^* + \sum_{j=1}^{3} i_j x_j^*$.
Then $f : \Omega \to \mathbb{H}$ is called left quaternionic differentiable at z^* if there exist three $C^0(\Omega)$-functions $\Psi_1, \Psi_2, \Psi_3 : \Omega \to Pu(\mathbb{H})$ satisfying at each z^* the relation $<\Psi_j(z^*), \Psi_k(z^*)>= \delta_{jk}$ and such that*

$$\lim_{\Delta z^{[\Psi]} \to 0} (\Delta z^{[\Psi]})^{-1}(\Delta f) \tag{6}$$

*exists. Hereby $\Delta z^{[\Psi]} = \Delta x_0 + \sum_{i=1}^{3} \Delta x_i \Psi_i(z^*)$ with $\Delta x_k = x_k - x_k^*$.
f is called left quaternionic differentiable in Ω if f is left quaternionic differentiable at every point $z \in \Omega$.
In an analogous way, right quaternionic differentiability may be defined for $f : \Omega \to \mathbb{H}$, namely by requiring that for each $z \in \Omega$, there exist three $C^0(\Omega)$-functions $\Phi_1, \Phi_2, \Phi_3 : \Omega \to Pu(\mathbb{H})$ such that $<\Phi_j(z), \Phi_k(z)> = \delta_{jk}$ and*

$$\lim_{\Delta z^{[\Phi]} \to 0} (\Delta f)(\Delta z^{[\Phi]})^{-1} \tag{7}$$

exists.

The limit

$$\lim_{\Delta z^{[\Psi]} \to 0} (\Delta z^{[\Psi]})^{-1}(\Delta f)$$

may be considered as a linearization of the function f at the point z^* with respect to the orthonormal basis $[1, \Psi_1(z^*), \Psi_2(z^*), \Psi_3(z^*)]$ and it is equal to the expression $\frac{\partial f}{\partial x_0}(z^*)$, which may be regarded as the left quaternionic derivative of f at the point z^*.

Notice that a C^1-function f is left quaternionic differentiable in Ω if and only if

$$\frac{\partial f}{\partial x_k}(z) = \Psi_k(z)\frac{\partial f}{\partial x_0}, \qquad k = 1, 2, 3, \tag{8}$$

where $\Psi_1(z), \Psi_2(z), \Psi_3(z)$ are C^0-functions being mutually orthonormal to each other at each point of the domain Ω.
As will be seen, non-constant quaternionic differentiable maps are nothing else but conformal maps in the sense of Gauss.

Definition 3. *Let $\Omega \subset \mathbb{H}$ be a domain. A real differentiable function $f : \Omega \to \mathbb{H}$ is called conformal in the sense of Gauss, if there exists a strictly positive real valued continuous function λ on $\mathbb{H}$, $z \mapsto \lambda(z)$, such that*

$$< df, df >= \lambda(z) < dz, dz > . \tag{9}$$

As in complex analysis we have

Theorem 1. *The set of left quaternionic differentiable functions coincides with the set of right quaternionic differentiable functions. A quaternionic differentiable function is either a conformal map in the sense of Gauss or a constant map.*

In contrast to the complex case the set of conformal mappings in higher dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 3$) is - by the famous theorem of Liouville (cf. [26, 17]) - restricted to the set of Möbius transformations.
Therefore, a C^1-function is quaternionic differentiable if and only if f is a linear fractional function, i.e. if there exist four quaternions a, b, c, d such that

$$f(z) = (az + b)(cz + d)^{-1}.$$

According to e.g. G. Zöll [56] we know that the Möbius transformation f can also be written in the form

$$f(z) = (z\gamma + \delta)^{-1}(z\alpha + \beta)$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{H}$ are - up to a non-zero real factor - uniquely defined by the coefficients a, b, c, d.

Remarks:

1. From Theorem 1 it thus follows that with a quaternionic differentiable function two structural sets $[\Psi]$ and $[\Phi]$ may be associated such that for $k = 1, 2, 3$

$$\frac{\partial f}{\partial x_k}(z) = \Psi_k(z)\frac{\partial f}{\partial x_0}, \quad k = 1, 2, 3 \tag{10}$$

and

$$\frac{\partial f}{\partial x_k}(z) = \frac{\partial f}{\partial x_0}\Phi_k(z), \quad k = 1, 2, 3. \tag{11}$$

In general, $[\Psi] \neq [\Phi]$.

2. Furthermore, notice that if f is a linear fractional function with $c \neq 0$, then with each point a different structural set is associated. Only if f has the form $f(z) = (az + b)d^{-1}$ (i.e. if $c = 0$), then the associated functions Ψ_i are constant functions which in general are different from the canonical structural set $[1, i_1, i_2, i_3]$. For details see [22].

3. If one restricts himself to the constant standard structural set, i.e. if we set $\Psi_j \equiv i_j$, then precisely functions of the form $f(z) = az + b$ or $f(z) = za + b$ are obtained. The use of structural sets thus provides a significant extension of the strict definition of differentiability given in (2) or (3).

4. According to [37] a function $f(z) = az + b$ where $a \in \mathbb{H}\backslash\mathbb{R}$ is only left differentiable with derivative equal to a, but not right differentiable in the sense of (3). However, Theorem 1 tells us that every left quaternionic differentiable function in the more general sense of Definition 2 is also right quaternionic differentiable with the same derivative. One could think that this result yields a contradiction but it does not, since we are working with structural sets. Indeed each $f(z) = az + b$ where $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ can be written in the form $f(z) =$

$z^{[\Psi]}a+b$ where $z^{[\Psi]} = x_0 + x_1\Psi_1 + x_2\Psi_2 + x_3\Psi_3$ is another structural set. So f is also right quaternionic differentiable with the same derivative, but as already mentioned, with respect to another orthonormal basis. Only when a is real, one obtains $\Psi_j = i_j$ whence we are in the case treated by J. J. Buff.

5. The paper [3] is also dealing in part with the problem of this section. However, the authors arrived at a wrong conclusion, namely that a function f is conformal in the domain Ω if and only if there is a constant structural set $[\Psi]$ such that

$$df(z) = (dx_0 + \Psi_1 dx_1 + \Psi_2 dx_2 + \Psi_3 dx_3)\frac{\partial f}{\partial x_0}(z) \tag{12}$$

holds for all $z \in \Omega$.
They have not observed that in general the structural sets are C^0-functions which take different values at each point of the domain. Furthermore, they did not establish an explicit relation between (12) and a condition of differentiability.
The article [22] provides therefore in some sense a rectification and completion of [3].

3 Hypercomplex differentiability and derivability

In Section 2 we saw that quaternionic differentiability characterizes the class of linear fractional functions. The fact that the algebra of real quaternions is a skew field was essential for this type of differentiability.
Now we shall be concerned with some kinds of differentiability for a class of Clifford algebra valued functions, the so-called class of monogenic functions.
As has already been pointed out in Section 1, $\mathbb{C}$ and $\mathbb{H}$ are special cases of real Clifford algebras, but in general Clifford algebras contain zero-divisors. However, just as in the quaternionic case, a Riemann approach to a function theory in Clifford algebras could be sucessfully developed. The question if a Cauchy approach is possible remained an open problem for a long time.
The fact that a Clifford algebra in general is not a division algebra suggested the idea to define differentiability in this case by the local linear approximation property without looking for some equivalent differential quotient as has been done in Section 2 in the case of quaternions. The consequence of such an approach (cf. [31]) was that it was not possible to obtain an equivalent for the complex derivative of a holomorphic function in the form as it would normally be expected. Indeed, whereas the derivative of a holomorphic function is again a holomorphic function, it follows from the approach mentioned that the derivative has the form of a vector of monogenic functions. However, recently the notion of derivability for Clifford algebra valued functions has been introduced in [15] and within that framework, monogenicity of a function is equivalent to the existence of a "derivative". The result thus obtained confirmed in some sense a remark made by S. Semmes in the problem section of [45] (page 25) where he noticed:

> I like to think of Clifford analysis (and other Riesz systems) as being "codimension-1 complex analysis" on $\mathbb{R}^n$.

Indeed, the consideration of certain relations between differential forms in codimension-1 (or, equivalently, the validity of special relations involving integrals over hypersurfaces) permits to show that every monogenic function possesses locally a (monogenic) derivative and that a Clifford algebra valued function over $\mathbb{R}^{n+1}$ with derivative (i.e. a *derivable* function) is a monogenic one.

Subsection 3.1 studies linear mappings and the corresponding linear approximation property (leading to the definition of *differentiability*) whereas the second subsection considers the question of the existence of a derivative (leading to the definition of *derivability*) of a Clifford algebra valued function in $\mathbb{R}^{n+1}$. Of course, in classical complex analysis, both concepts are equivalent. This is not anymore the case in the general Clifford algebra setting. Indeed, differentiability is related to 1-forms while derivability is related to n-forms.

3.1 Local linear approximation of $C\ell_{0,n}$-valued functions defined in $\mathbb{R}^{n+1}$

Let the vector space $\mathbb{R}^n$ be endowed with a non-degenerate bilinear form of signature $(0,n)$ and let $C\ell_{0,n}$ be the universal 2^n-dimensional real Clifford algebra constructed over $\mathbb{R}^{0,n}$. According to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl} e_0, \quad k,l = 1,\ldots,n,$$

the set $\{e_A : A \subseteq \{1,\ldots,n\}\}$ with $e_A = e_{h_1}e_{h_2}\cdots e_{h_r}, 1 \leq h_1 < \ldots < h_r \leq n, \quad e_\emptyset = e_0 = 1$, is a basis of $C\ell_{0,n}$. The conjugate $\bar{\alpha}$ of $\alpha = \sum_A \alpha_A e_A \in C\ell_{0,n}$ is given by $\bar{\alpha} = \sum_A \alpha_A \bar{e}_A$ where $\alpha_A \in \mathbb{R}$ and $\bar{e}_A = \bar{e}_{h_r}\bar{e}_{h_{r-1}}\cdots\bar{e}_{h_1}$, with $\bar{e}_k = -e_k$ $(k = 1,\ldots,n)$ and $\overline{e_0} = e_0 = 1$.
Identifying each element $x = (x_0, x_1, \ldots, x_n)$ of $\mathbb{R}^{n+1}$ with

$$z = x_0 + x_1 e_1 + \ldots + x_n e_n \in \mathcal{A} := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\}$$

the conjugate of z is given by

$$\bar{z} = x_0 - \sum_{k=1}^{n} x_k e_k.$$

The norm of $z \in \mathcal{A}$ is $|z| := \sqrt{z\bar{z}}$. Like in $\mathbb{H}$, it immediately follows that each $z \in \mathcal{A}\backslash\{0\}$ is invertible and its inverse is

$$z^{-1} = \frac{\bar{z}}{|z|^2}.$$

In what follows we consider $C\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$, i.e.: functions of the form

$$f(z) = \sum_A f_A(z) e_A,$$

where the $f_A(z)$ are real valued and we regard them as mappings

$$f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{A} \mapsto C\ell_{0,n}. \tag{13}$$

Historically (cf. Chapter 2 of [2]), Clifford algebra valued functions generalizing the holomorphic functions were defined as elements of the kernel of the Cauchy-Riemann operator D in $\mathbb{R}^{n+1}, n \geq 1$, given by

$$D = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + \ldots + e_n\frac{\partial}{\partial x_n}. \tag{14}$$

Functions f satisfying the equation $Df = 0$ are called *left monogenic* and solutions of $fD = 0$ are called *right monogenic*. In general, only left monogenic functions are considered, the theory of right monogenic functions being analogous.
The theory of functions related to the Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is nowadays commonly called Clifford analysis. The conjugate Cauchy-Riemann operator is defined by

$$\overline{D} = \frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - \ldots - e_n\frac{\partial}{\partial x_n}, \tag{15}$$

thus providing the factorization $D\overline{D} = \overline{D}D = \Delta_{n+1}$ where Δ_{n+1} is the Euclidean Laplacian in $\mathbb{R}^{n+1}$. Some basic examples of monogenic functions are

$$z_k = x_k - x_0 e_k; \quad x_0, x_k \in \mathbb{R}, \quad k = 1, \ldots n.$$

These functions are totally regular in the sense of [6] since $(z_k)^m$, $m \in \mathbb{N}$, is still monogenic. Unfortunately, the same is not true for $z \in \mathcal{A}$ - neither z nor z^n belong to the kernel of D. Nevertheless, it should be noticed that z^k $k \in \mathbb{N}$, are null-solutions of $D\Delta^m_{2m+2}$ in $\mathbb{R}^{2m+2}$ (see [25]).
The following representation of the increment of Δf of f in a neighbourhood of $z \cong x \in \mathbb{R}^{n+1}$ shows the special role of the totally regular functions z_k, $k = 1, \ldots, n$.
In the sequel we put $\vec{z} = (z_1, z_2, \ldots, z_n)$. Notice that for each $z = x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathcal{A}$ and $k = 1, \ldots, n$

$$z_k = -\frac{ze_k + e_k z}{2},; \quad k = 1, \ldots, n.$$

Some formal coincidence with the corresponding Cauchy approach to holomorphic functions of several complex variables suggests to introduce a norm of $\vec{z}$ by means of

$$\|\vec{z}\| := < \vec{z}, \overline{\vec{z}} >^{\frac{1}{2}} = (z_1\bar{z}_1 + \ldots + z_n\bar{z}_n)^{\frac{1}{2}} = (nx_0^2 + x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}.$$

Suppose that f is real differentiable at z, then (cf. [31]):

$$\begin{aligned}
\Delta f(z) &= \frac{\partial f}{\partial x_0}\Delta x_0 + \frac{\partial f}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n + o(|\Delta x|) \\
&= \Delta x_0 D f + \\
&\quad +(\Delta x_1 - e_1\Delta x_0)\frac{\partial f}{\partial x_1} + \cdots + (\Delta x_n - e_n \Delta x_0)\frac{\partial f}{\partial x_n} + o(|\Delta x|) \\
&= \Delta z_0 D f \\
&\quad +\Delta z_1 \frac{\partial f}{\partial x_1} + \cdots + \Delta z_n \frac{\partial f}{\partial x_n} + o(\|\Delta \vec{z}\|))
\end{aligned} \tag{16}$$

where

$$\lim_{\Delta x \to 0} \frac{o(|\Delta x|)}{|\Delta x|} = \lim_{\Delta \vec{z} \to 0} \frac{o(\|\Delta \vec{z}\|)}{\|\Delta \vec{z}\|} = 0. \tag{17}$$

Besides the assumption of real differentiability we hereby used the transformation

$$z_0 := x_0, \text{ and } \quad z_k := x_k - x_0 e_k, \quad k = 1, \ldots, n,$$

with inverse

$$x_0 = z_0, \quad x_k = z_0 e_k + z_k, \quad k = 1, \ldots, n.$$

Monogenicity of f, i.e. $Df = 0$, implies now that the differential of f in (16) becomes approximately a linear combination only in n increments and this fact is the key to the desired concept of hypercomplex differentiability which should characterize the class of monogenic functions in the way complex differentiability characterizes holomorphic functions.
Indeed, following [31] and the previous lines of reasoning we are led to consider the isomorphism

$$\mathbb{R}^{n+1} \cong \mathcal{H}^n = \{\vec{z} : z_k = x_k - x_0 e_k; x_0, x_k \in \mathbb{R}\}.$$

It defines a second hypercomplex structure, different from that given by $\mathcal{A} = \mathcal{A}_n$. The similarity with $\mathbb{C}^n$ and at the same time their differences become clear by considering n copies $\mathbb{C}_k$ of $\mathbb{C}$ identifying $i \cong e_k$, $(k = 1, \ldots, n)$; $x_0 \cong \Re z$; $x_k \cong \Im z$; where $z \in \mathbb{C}$, and taking then $\mathbb{C}_k := -e_k\mathbb{C}$. We obtain

$\mathcal{H}^n$ as the cartesian product $\mathcal{H}^n := \mathbb{C}_1 \times \cdots \times \mathbb{C}_n$.
From this point of view, a mapping $f \in C^1(\Omega; C\ell_{0,n})$ may be regarded as a mapping from $\mathcal{H}^n$ into $C\ell_{0,n}$ and we may ask for the general form of $C\ell_{0,n}$-linear mappings from $\mathcal{H}^n$ into $C\ell_{0,n}$.
In this context it is important to point out that $\mathcal{H}^n$ is a special subset of the n-fold cartesian product $(C\ell_{0,n})^n$ of $C\ell_{0,n}$ but not a $C\ell_{0,n}$-submodule since $\lambda\vec{z}$, or $\vec{z}\lambda$ belong to $\mathcal{H}^n$ if and only if $\lambda \in \mathbb{R}$. Nevertheless the imbedding of $\mathcal{H}^n$ in the module $(C\ell_{0,n})^n$ enables us to use the properties of $(C\ell_{0,n})^n$ for describing $C\ell_{0,n}$-linear mappings from $\mathcal{H}^n$ into $C\ell_{0,n}$ in the following way (cf. [31]):

Theorem 2. *A left $C\ell_{0,n}$-linear mapping $\ell_L \in \mathcal{L}(\mathcal{H}^n; C\ell_{0,n})$ may be represented as*

$$\ell_L(\vec{z}) = z_1 A_1 + \cdots + z_n A_n$$

where $A_k \in C\ell_{0,n}$, $(k = 1, \ldots, n)$.
In an analogous way, a right $C\ell_{0,n}$-linear mapping $\ell_R \in \mathcal{L}(\mathcal{H}^n; C\ell_{0,n})$ may be represented as

$$\ell_R(\vec{z}) = \tilde{A}_1 z_1 + \cdots \tilde{A}_n z_n,$$

where $\tilde{A}_k \in C\ell_{0,n}$, $(k = 1, \ldots, n)$.
Moreover, the elements A_k and $\tilde{A}_k$ are unique.

Equipped with the general form of the linear mappings $\ell_L(\ell_R) \in \mathcal{L}(\mathbb{H}^n; C\ell_{0,n})$, the following definition ([30],[31]) now seems very natural:

Definition 4. *Let f be a continuous mapping from a neighborhood of $\vec{z} \in \mathcal{H}^n$ into $C\ell_{0,n}$. Then f is called left hypercomplex differentiable (resp. right hypercomplex differentiable) at $\vec{z}$ if there exists a left (resp. right) $C\ell_{0,n}$-linear mapping ℓ such that*

$$\lim_{\Delta\vec{z}\to 0} \frac{|f(\vec{z}+\Delta\vec{z}) - f(\vec{z}) - \ell(\Delta\vec{z})|}{\|\Delta\vec{z}\|} = 0. \tag{18}$$

We say that a function f is hypercomplex differentiable in $\Omega \subset \mathbb{R}^{n+1} \cong \mathcal{H}^n$ if it is hypercomplex differentiable at all points of Ω.

Obviously, the relation (18) implies that the differential of a left (resp. right) hypercomplex differentiable function has the form

$$df = dz_1 \frac{\partial f}{\partial x_1} + \cdots + dz_n \frac{\partial f}{\partial x_n} \tag{19}$$

$$\text{resp. } df = \frac{\partial f}{\partial x_1} dz_1 + \cdots + \frac{\partial f}{\partial x_n} dz_n. \tag{20}$$

So, we have

Theorem 3. *If $f(\vec{z})$ is hypercomplex differentiable then the corresponding linear mapping ℓ_L (resp. ℓ_R) (also called the L resp.(R-) derivative) is determined in a unique way.*

The following theorem shows that the concept of hypercomplex differentiability represents the Cauchy approach to the theory of monogenic functions.

Theorem 4. *Let $f = f(\vec{z})$ be continuously real differentiable in an open set $\Omega \subset \mathcal{H}^n$. Then f is hypercomplex L- (R-) differentiable in Ω, if and only if f is L- (R-) monogenic in Ω.*

Let us illustrate this in $\mathbb{R}^3 \equiv \mathcal{A}_2 \equiv \mathcal{H}^2$ by the following example. Take the monogenic polynomial

$$h(x) = x_1 x_2 - x_0 x_2 e_1 - x_0 x_1 e_2. \tag{21}$$

Then h may also be expressed as

$$\begin{aligned} h(x) &= \frac{1}{8}[(ze_1e_2z + e_1ze_2z + e_1ze_2z + e_1zze_2) \\ &\quad + (ze_2e_1z + e_2ze_1z + e_2ze_1z + e_2zze_1)] \\ &= \frac{1}{2}(z_1z_2 + z_2z_1) \end{aligned}$$

The last expression allows a simple direct calculation of the increment and its linearization:

$$\begin{aligned} \frac{1}{2}[(z_1 + \Delta z_1)(z_2 + \Delta z_2) + (z_2 + \Delta z_2)(z_1 + \Delta z_1) - (z_1z_2 + z_2z_1)] &= \\ \frac{1}{2}[(\Delta z_1 \cdot z_2 + z_1 \cdot \Delta z_2 + \Delta z_2 \cdot z_1 + z_2 \cdot \Delta z_1) + (\Delta z_1 \Delta z_2 + \Delta z_2 \Delta z_1)] &= \\ z_1 \cdot \Delta z_2 + z_2 \cdot \Delta z_1 + o(||\Delta \vec{z}||) &= \\ \Delta z_1 \cdot z_2 + \Delta z_2 \cdot z_1 + o(||\Delta \vec{z}||). \end{aligned}$$

Hereby, we used the property that

$$\Delta z_1 \cdot z_2 + z_1 \cdot \Delta z_2 = \Delta z_2 \cdot z_1 + z_2 \cdot \Delta z_1.$$

Finally notice that neither the product $z_1 \cdot z_2$ nor $z_2 \cdot z_1$ permits a representation of the increment as a pure left- resp. right-linear expression, because they are not monogenic. This fact sheds some light on the contrast between $\mathbb{C}^n$ and $\mathcal{H}^n$ which is caused by the non-commutativity of the Clifford algebra.

3.2 The codimension-1 case: monogenic derivatives

From (19) it is clear that in the case of monogenic functions the derivative will be given by the gradient of f with respect to $\vec{x} = (x_1, \ldots, x_n)$. Of course, this is a vector of monogenic functions which defines f up to a constant [31]. Notice that in the complex case

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \frac{1}{2}\bar{D}f = \frac{1}{2}\Big(\frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1}\Big) = -i\frac{\partial f}{\partial x_1},$$

while in our case, for $n = 1$ we obtain $\frac{\partial f}{\partial x_1}$.
This corresponds to the fact that, by identifying e_1 by i,

$$z_1 = x_1 - x_0 e_1 = \overline{e_1}(x_0 + e_1 x_1).$$

However, for $n > 1$, hypercomplex differentiability does not clarify the role of $\overline{D}f$ as a possible candidate for the "derivative" of a monogenic function. Let us now describe how this can be realized. As far as we know A. Sudbery was the first who noticed in [51] that in the case of a monogenic quaternion valued function f the exterior derivative of a special 2-form is equal to the quaternionic surface element multiplied by $\bar{D}f$. As is usual in the theory of differential forms such a multiplier is called a differential coefficient (cf. [8]). In this sense A. Sudbery explicitly considered $\bar{D}f$ as the derivative of f and gave a characterization in terms of limits of partial increments of f.
Notice that the calculus of non-alternating quaternionic differential forms used by A. Sudbery was some years later generalized to the case of special monogenic functions by J. Ryan [44] and this by using tensor calculus methods and the Hodge star operator, without mentioning $\overline{D}$ explicitely nor its

possible interpretation as a monogenic derivative. A general theory of differential forms in the framework of Clifford analysis, including homology theory and topics related to integral theorems in $\mathbb{C}^n$ may be found in [47], [48] or [43].
Following A. Sudbery's idea our aim is to show how the hypercomplex derivative of a (left) monogenic function f can be defined as the (left) differential coefficient of two differential forms of degree n.
As before consider a real differentiable function $f : \Omega \mapsto C\ell_{0,n}$, where $\Omega \subset \mathcal{A}$ is open. Its differential at a point $z \in \mathcal{A}$ is then an $\mathbb{R}$-linear mapping and a $C\ell_{0,n}$-valued 1-form $df : \mathcal{A} \mapsto C\ell_{0,n}$:

$$df = \frac{\partial f}{\partial x_0}dx_0 + \frac{\partial f}{\partial x_1}dx_1 \cdots + \frac{\partial f}{\partial x_n}dx_n. \tag{22}$$

The differential forms needed for our purpose are elements of the real vector space of $\mathcal{A}$-valued p-forms $\bigwedge_{\mathcal{A}}^p$, with a basis of real p-forms consisting of exterior products of the real 1-forms $dx_k,\ \ k = 0, \ldots, n$. The differential of the identity function $f(z) = z$ given by

$$dz = dx_0 + e_1 dx_1 \cdots + e_n dx_n \tag{23}$$

will play an essential role. Applying to $\mathcal{A}$-valued forms the well known Hodge star-operator $*$, a linear transformation between the pairs of spaces $\bigwedge_{\mathcal{A}}^p$ and $\bigwedge_{\mathcal{A}}^{(n+1)-p}$, is obtained, i.e.

$$* : \bigwedge\nolimits_{\mathcal{A}}^p \longrightarrow \bigwedge\nolimits_{\mathcal{A}}^{(n+1)-p}, \quad p = 0, \ldots, (n+1).$$

In such way for $f \equiv 1 \in \bigwedge_{\mathcal{A}}^0$

$$*1 = dV|_{\mathbb{R}^{n+1}} := dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n,$$

is the volume element in $\mathbb{R}^{n+1}$, while for $dz = dx_0 + e_1 dx_1 \cdots + e_n dx_n$ we obtain

$$*dz = d\sigma_{(n)} := d\hat{x}_0 - e_1 d\hat{x}_1 + \cdots + (-1)^n e_n d\hat{x}_n.$$

Hereby, as usual,

$$d\hat{x}_j = dx_0 \wedge \cdots \wedge [dx_j] \wedge \cdots \wedge dx_n.$$

Notice that for $n = 1$,

$$*dz = dx_1 - i dx_0 = -i dz = dz_1$$

and that (cf. [32]) in the setting of $\mathcal{H}^n$

$$d\sigma_{(n)} := dz_1 \wedge \cdots \wedge dz_n.$$

Now consider the 1-form

$$\text{Pu}\, dz = e_1 dx_1 + \ldots + e_n dx_n.$$

This 1-form is related to a hyperplane $x_0 = c,\, c \in \mathbb{R}$, and can be considered as a 1-form in $\bigwedge_{\mathcal{A}|_{x_0=c}}^1 \cong \bigwedge_{\mathbb{R}^n}^1$. With respect to the variables $(x_1, \ldots, x_n)$ the corresponding volume-form is given by

$$dV|_{\mathbb{R}^n} = dx_1 \wedge \cdots \wedge dx_n,$$

and the corresponding surface-element is

$$*(\text{Pu}\, dz) = d\sigma_{(n-1)} := -e_1 d\hat{x}_{0,1} + e_2 d\hat{x}_{0,2} + \cdots + (-1)^n e_n d\hat{x}_{0,n}. \tag{24}$$

Here the Hodge star-operator has been applied to $\bigwedge_{\mathbb{R}^n}^1$ and the notation $d\hat{x}_{0,m}\ \ (m = 1, \ldots, n)$ means that in the ordered outer product of the 1-forms $dx_k\ \ (k = 0, \ldots, n)$ the factors dx_0 and dx_m are

absent.
Notice that for $n = 1$ we obtain the constant 0 - form $d\sigma_{(0)} = -e_1 \cong -i$, i.e. exactly the factor that is necessary to pass from $z \in \mathbb{C}$ to $z_1 = -iz \in \mathbb{C}_1$.

Now let $\nu = (\nu_0, \nu_1, \ldots, \nu_{p-1})$, $0 \leq \nu_0 < \nu_1 < \cdots < \nu_{p-1} \leq n$, be a multi-index. Every basic p-form $\omega_p \in \bigwedge_{\mathcal{A}}^p$ can be written in a unique way as

$$\omega_p = dx_{\nu_0} \wedge dx_{\nu_1} \wedge \ldots \wedge dx_{\nu_{p-1}} = dx_\nu.$$

Furthermore, for a given set $f_\nu = f_\nu(z)$ of $\begin{pmatrix} n+1 \\ p \end{pmatrix}$ $C\ell_{0,n}$-valued continuous functions in $\Omega \subset \mathcal{A}$ open, $\omega_p = \sum_\nu dz_\nu f_\nu(z)$, resp. $\omega_p = \sum_\nu f_\nu(z) dz_\nu$, is called a left, resp. a right, $C\ell_{0,n}$-valued p-form.

Definition 5. *Let*

$$\omega_p = \sum_\nu dz_\nu f_\nu(z)$$

be a continuously real differentiable left p-form on $\Omega \subset \mathcal{A}$. Then its derivative $d\omega_p$ is defined as the $(p+1)$-form

$$d\omega_p = \sum_\nu (-1)^p dz_\nu \wedge df_\nu(z)$$

where df_ν is the differential (22) of f_ν, i.e. the outer derivative of the 0-form f_ν.

For right p-forms or two-sided p-forms ω_p (i.e. having $C\ell_{0,n}$-valued coefficients on both sides) the definition of $d\omega_p$ is straightforward.
The significance of the special differential forms $d\sigma_{(n)}$ and $dV|_{\mathbb{R}^{n+1}}$ now becomes immediately clear. Indeed, consider the n-form $\omega = d\sigma_{(n)} f$. Then

$$\begin{aligned} d\omega &= (-1)^n d\sigma_{(n)} \wedge df \\ &= (-1)^n (d\hat{x}_0 - e_1 d\hat{x}_1 + \cdots + (-1)^n e_n d\hat{x}_n) \wedge (dx_0 \frac{\partial f}{\partial x_0} + dx_1 \frac{\partial f}{\partial x_1} + \cdots \\ &\quad \cdots + dx_n \frac{\partial f}{\partial x_n}) \\ &= dV(Df), \end{aligned}$$

whence ω is closed if $Df = 0$, i.e. if f is monogenic.

Let us now introduce the notion of hypercomplex derivability.

Definition 6. *A function $f : \mathcal{A} \mapsto C\ell_{0,n}$ is left derivable at $z \in \mathcal{A}$ if it is real differentiable at z and there exists $A_{f,L}(z) \in C\ell_{0,n}$ such that*

$$d(d\sigma_{(n-1)} f) = d\sigma_{(n)} A_{f,L}(z). \tag{25}$$

Analogously, a real differentiable function f at $z \in \mathcal{A}$ is right derivable if there exists $A_{f,R}(z) \in C\ell_{0,n}$ such that

$$d(f d\sigma_{(n-1)}) = A_{f,R}(z) d\sigma_{(n)}. \tag{26}$$

$A_{f,L}(z)$, respectively $A_{f,R}(z)$, are called the left, resp. right, derivative of f at z.

We have

Theorem 5. *A real differentiable function $f : \Omega \subset \mathcal{A} \to \mathcal{C}\ell_{0,n}$ is left derivable if and only if $Df(z) = 0$, $z \in \Omega$, i.e. f is left monogenic in Ω.*

The proof relies on the relation

$$d(d\sigma_{(n-1)}f) = (-1)^{n-1} d\sigma_{(n-1)} \wedge df = \frac{1}{2} d\sigma_{(n)} \overline{D} f - \frac{1}{2} \overline{d\sigma_{(n)}} D f \quad (27)$$

Theorem 5 motivates the treatment of $\frac{1}{2}\overline{D}f$ as the derivative $f'_L(z)$ of the monogenic function f. Indeed, as we have seen in the case $n = 1$, putting $e_1 = i$, we have that

$$\frac{1}{2}\overline{D}f = -i\frac{\partial f}{\partial x_1}.$$

That for f monogenic in Ω, $\frac{1}{2}\overline{D}f$ is monogenic in Ω is a simple consequence of $D\overline{D}f = \overline{D}Df = 0$ in Ω.
Finally, notice that by Stokes' theorem it is also possible to express the hypercomplex derivative $\frac{1}{2}\overline{D}f$ as the limit of a quotient of two integrals.
Indeed, suppose f is monogenic in Ω and $\mathcal{S} \subset \Omega$ is an oriented differentiable n-dimensional hypersurface $\mathcal{S}$ with boundary $\partial\mathcal{S}$. From (27) it follows that

$$d(d\sigma_{(n-1)}f) = d\sigma_{(n)}\frac{1}{2}\overline{D}f \quad (28)$$

whence by Stokes' theorem

$$\int_{\partial\mathcal{S}} (d\sigma_{(n-1)}f) = \int_{\mathcal{S}} d\sigma_{(n)}\frac{1}{2}\overline{D}f. \quad (29)$$

Now take $z^* \in \mathcal{S}$ and consider a sequence of subdomains $\{\mathcal{S}_m\}$ which is shrinking to z^* if $m \to \infty$. Then

$$\frac{1}{2}\overline{D}f = \lim_{m\to\infty} \left[\int_{\mathcal{S}_m} d\sigma_{(n)}\right]^{-1} \int_{\partial\mathcal{S}_m} (d\sigma_{(n-1)}f). \quad (30)$$

It thus follows that the nature of the derivative $\frac{1}{2}\overline{D}$ is that of an areolar derivative (cf. [52] and [53]).

4 Final remarks

As was already mentioned in the very beginning of the paper, finding an appropriate definition of differentiability of functions with values in some real associative algebras and building up a related function theory, was a research topic already dealt with more than a century ago (see for instance R. W. Hamilton and later on A. C. Dixon (cf. [19, 7])). The lack of a satisfactory answer - already in the case of the algebra of real quaternions - led G. C. Moisil, N. Théodoresco, R. Fueter and successors to concentrate on other approaches to function theory.
Nevertheless, many attempts were still made to solve the differentiability problem for such algebra-valued functions defined in higher dimensional Euclidean space. In the previous sections we pointed out some of them.
The paper [16] is a good reference for early results on function theory in algebras and describes other attempts for defining a meaningful concept of differentiability (different from those we mentioned and not leading to linear fractional or monogenic functions in the case of Clifford algebras.)
The study of functions of a quaternionic variable via the representation of real quaternions by pairs of complex numbers has also been developed (cf. [40], [35], [36], [41]).

A concept of hyperderivability was given in [38]. It leads indirectly to a justification of the conjugate Cauchy-Riemann operator as representing the quaternionic derivative (like it was done by A. Sudbery before in [51]).
V. Souček also started his paper [49] with the question of differentiability, but his aim of splitting $\mathbb{H}$-valued differential forms over $\mathbb{H}$ with the help of the Dirac operator as well as the Twistor operator confirmed that quaternionic differentiable and hypercomplex differentiable functions are important and, in some sense, complementary classes of functions.
Also motivated by physical problems like in the case of V. Souček is the view on the differentiability problem explained by K. Imaeda in [20]. The special feature of his Lecture Notes is the consideration of biquaternions and the interpretation of the hypercomplex derivative as a functional derivative which is only a former term for areolar derivative.
This leads us again to Pompeiu's notion of *dérivée aréolaire* ([42]) which we already mentioned at the end of Subsection 3.2.
The original definition deals with domains in $\mathbb{R}^2$ and reads as follows.

Let z^* be a fixed point in a domain $G \subset \mathbb{C}$ where G is bounded by a finite number of piece-wise smooth Jordan curves. Consider a so called regular sequence of subdomains $\{G_n\}$ which is shrinking to z_* if n tends to infinity and whereby z_* belongs to all G_n. If for some function w in G,

$$\lim_{n\to\infty} \frac{1}{\text{mes } G_n} \frac{1}{2i} \int_{\partial G_n} w(z)dz \tag{31}$$

exists, then this limit is called the areolar derivative of $w = w(z)$ at z_*.

Pompeiu's motivation for considering such type of derivative came from measure theory (cf. [52]) and in this setting only weak smoothness conditions on w were required.

Now observe that the combination of Stokes' theorem for C^1- functions

$$\frac{1}{2i} \int_{\partial G} w(z)d\bar{z} \quad = \quad -\iint_G \partial_z w dx\, dy, \tag{32}$$

and the two-dimensional version of the mean value theorem for Lebesgue integrals

$$\lim_{n\to\infty} \frac{1}{\text{mes } G_n} \iint_{G_n} f(z)dx\, dy \quad = \quad f(z^*)$$

leads directly to

$$\lim_{n\to\infty} \left[-\frac{1}{\text{mes } G_n} \frac{1}{2i} \int_{\partial G_n} w(z)d\bar{z} \right] \quad = \quad \partial_z w(z^*), \tag{33}$$

which is exactly a relation of the form (31). It shows that $\partial_z w$ is an areolar derivative (but also $\partial_{\bar{z}} w$ as we can see by substitution of $\bar{z}$ by z in Stokes' theorem). But, as

$$\text{mes } G = \int_G dxdy = -\frac{1}{2i} \int_{\partial G} dz \wedge d\bar{z} \tag{34}$$

we see that $2i$ mes G_n in (33) can be substituted by $-\int_{\partial G_n} dz \wedge d\bar{z}$. Hence

$$\lim_{n\to\infty} \left[\frac{1}{\int_{\partial G_n} dz \wedge d\bar{z}} \int_{\partial G_n} w(z)d\bar{z} \right] \quad = \quad \partial_z w. \tag{35}$$

It is easy to verify that (35) may be derived from (30) in the case $n = 1$. This justifies to call $\frac{1}{2}\overline{D}f$ an aerolar derivative in the sense of Pompeiu, as we did in Subsection 3.2.

However, notice that in the higher dimensional case the integral $\int_{S_m} d\sigma_{(n)}$ is not a product of mes S_m with a constant Clifford number. The complex case is therefore exceptional, since $d\sigma_{(0)} = -e_1 \cong -i$ is constant. The deeper reason for this phenomenon stams from the fact that in the case of $\mathbb{R}^2$, dimension 1 and codimension-1 coincide. And this is clearly also the reason why the complex derivative of a holomorphic function can be represented in two equivalent forms: as an ordinary derivative (i.e. ordinary differential quotient) and an areolar derivative. Notice also that, in order to obtain (30), we have put some smoothness conditions on the functions considered. This is mainly due to the fact that the measure theoretical relevance of Pompeiu's approach is not essential for the problem we are dealing with.

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