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Differentiability of functions with values in some real associative algebras: approaches to an old problem

by

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Abstract

During the last one hundred and fifty years several mathematicians and physicists considered the question of generalizing real or complex differentiability and derivability to the case of algebra-valued functions and their relation to higher dimensional function theories. In this paper we give an overview of some attempts and approaches.

1 General remarks and historical review

1.1 Origins of the problem

The classical definitions of differentiability and the differential as the linear part of the increment can be extended in several ways. The extension to functions defined on a real or complex Banach space is one of the most general ones. It leads to the concepts of differentiability in the sense of Gâteaux or Fréchet.

The concrete determination of the differential, i.e. the approximation of the function considered in a neighbourhood of some point by a linear mapping, is an important operation in differential calculus in finite dimensional real or complex vector spaces.
As an example we refer to the well-known fact that if $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, then the derivative of a real differentiable function $f : X \to Y$ is defined by the Jacobian matrix which is a continuous $\mathbb{R}$-linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$.
This type of derivative is a directional derivative: it is related to the fact that division by an element of the scalar field is possible.

Keywords: differentiability, derivability, algebra-valued functions

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