WAVELETS IN SOBOLEV SPACES

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Abstact. In the first part, we present two constructions of biorthogonal bases of wavelets in Sobolev spaces $H^m(\mathbb{R})$ of integer order. On one hand, we give a Sobolev version of the L^2 biorthogonal wavelets bases construction of A. Cohen, I. Daubechies, J.C. Feauveau ([15]), and on the other hand, a Sobolev version of the L^2 classical construction of C.K. Chui and J.Z. Wang ([13]).

In the second part, we show how hierarchical periodic spline spaces can be used to approximate solutions for a large class of pseudodifferential equations on boundaries of smooth open subsets of $\mathbb{R}^2$. We give a simple proof of the characterization of the coercivity condition, which leads to relations on the meshes and order of the splines easy to handle. Then we endow the periodic Sobolev space $H^s_{1-\mathrm{per}}(\mathbb{R})$ with a norm equivalent to the natural one which makes the Galerkin system of Céa lemma equivalent to a collocation system for high resolution levels. Test and trial spline bases are explicitly given. We investigate the asymptotic stability of such systems and we present some numerical experiments.

Key words : Wavelets, Sobolev spaces, spline approximation, collocation methods.
AMS Classification: 42C40, 46E35, 41A15, 65L60.

Introduction

As a general introduction to wavelets, let us quote I. Daubechies ([24]): *"There are several reasons for their success. On one hand, the concept of wavelets can be viewed as a synthesis of ideas which originated during the last twenty or thirty years in engineering (subband coding), physics (coherent states, renormalization group), and pure mathematics (study of Calderón-Zygmund operators) [...] On the other hand, wavelets are a fairly simple mathematical tool with a great variety of possible applications (signal analysis, numerical analysis, ...)"*. Wavelets represent an alternative to Fourier analysis; their typical property of localization and their hierarchical construction are the main reasons why they constitute a new way and tool to look at many problems.

Typically, an orthonormal basis of wavelets is an orthonormal basis of $L^2(\mathbb{R})$ generated by shifts and dilates of a single function, i.e.

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}.$$

In some applications, orthogonality can be felt as a too rigid requirement. Indeed, under a signal analysis point of view, orthogonality means that the signal has to be analyzed and reconstructed with the same filter. Moreover, exact reconstruction and symmetry are incompatible. In order to get more flexibility on the filters, orthonormal bases can be relaxed and replaced by Riesz bases and dual Riesz bases. Symmetry is then possible. Another important utility of Riesz bases is that they characterize many functional spaces by means of wavelets coefficients (see [24], [36]).

It is now clearly established that Sobolev spaces are the natural framework in which partial differential equations are well posed. This is why we are interested in this work to investigate bases of wavelets in those functional spaces.

Now let us introduce more specifically the contents of our work.

In her paper [23], I. Daubechies constructs orthogonal bases of wavelets in $L^2(\mathbb{R})$ with compact support and arbitrary high regularity. In [9], [10], F. Bastin and P. Laubin generalize these results to Sobolev spaces. They give a general procedure to construct orthogonal wavelets in these spaces and show how it is possible to obtain orthogonal compactly supported wavelets of arbitrary high regularity in Sobolev spaces of integer order $H^m(\mathbb{R})$. In their construction, the scaling function and the wavelets depend on the level j. The paper [37] of C. Micchelli contains general results on filters with applications to orthogonal wavelets in Sobolev spaces too.

Based on the papers [9] and [10], the first chapter of this work presents two constructions of biorthogonal bases of wavelets in $H^m(\mathbb{R})$ $(m \in \mathbb{N})$.

On one hand, we give a Sobolev version of the L^2 biorthogonal wavelets bases construction of A. Cohen, I. Daubechies, J.C. Feauveau ([15]). In [15], the authors construct wavelets $\psi, \widetilde{\psi}$ with compact support, arbitrary high regularity and symmetry axis; in their construction, the families $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ and $\{\widetilde{\psi}_{j,k}(x) = 2^{j/2}\widetilde{\psi}(2^j x - k) : j, k \in \mathbb{Z}\}$ are frames. They give a necessary and sufficient condition to obtain dual Riesz bases. Under fairly general conditions on filters, we construct here frames $\{\psi_{j,k}(x) = 2^{j/2}\psi^{(j)}(2^j x - k) : j, k \in \mathbb{Z}\}$, $\{\widetilde{\psi}_{j,k}(x) = 2^{j/2}\widetilde{\psi}^{(j)}(2^j x - k) : j, k \in \mathbb{Z}\}$ in $H^m(\mathbb{R})$. We show how it is possible to choose the filters to obtain regular real biorthogonal wavelets $\psi^{(j)}$, $\widetilde{\psi}^{(j)}$ with compact support independent of j and symmetry axis. These results are here carried out using procedures and filters of the papers [9] and [10]. In our construction, the functions $\psi^{(j)}(x/2)$ are exponential splines.

On the other hand, we give a Sobolev version of the L^2 classical construction of C.K. Chui and J.Z. Wang (see [13]). Remind that in [13], the functions $\psi(x/2), \widetilde{\psi}(x/2)$ are both spline functions, but only one of them has a compact support. The families $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ and $\{\widetilde{\psi}_{j,k} : j, k \in \mathbb{Z}\}$ are Riesz bases of $L^2(\mathbb{R})$ fulfilling the biorthogonality condition $\langle \psi_{j,k} , \widetilde{\psi}_{j',k'} \rangle = \delta_{k,k'}\delta_{j,j'}$. Moreover, in this case, the functions $\psi_{j,k}$ and $\psi_{j',k'}$ (resp. $\widetilde{\psi}_{j,k}$ and $\widetilde{\psi}_{j',k'}$) corresponding to different levels j and j' are orthogonal. In $H^m(\mathbb{R})$, we give filters that generate families of biorthogonal wavelets $\{\psi_{j,k}(x) = 2^{j/2}\psi^{(j)}(2^j x - k) : j, k \in \mathbb{Z}\}$, $\{\widetilde{\psi}_{j,k}(x) = 2^{j/2}\widetilde{\psi}^{(j)}(2^j x - k) : j, k \in \mathbb{Z}\}$ such that $\psi^{(j)}(x/2)$, $\widetilde{\psi}^{(j)}(x/2)$ are exponential splines, $\psi^{(j)}$ has a compact support independent of j and $\langle \psi_{j,k} , \psi_{j',k'} \rangle = 0$, $\langle \widetilde{\psi}_{j,k} , \widetilde{\psi}_{j',k'} \rangle = 0$ if $j \neq j'$.

Notice that both of our constructions presented in this first chapter can be interpreted as a procedure of exact reconstruction of signal. The differences come from nature and orthogonality of the constructed functions.

In order to apply these ideas to the resolution of partial differential equations and in particular to solve boundary value problems, the second chapter is devoted to the construction of wavelet Riesz bases in the Sobolev space $H^s_{1-\mathrm{per}}(\mathbb{R})$ of 1-periodic distributions, for every real number s.

For any strictly positive integer m and any $\ell \in \{1, \ldots, m\}$, we focus on the functions $\Psi^{(\ell)}_{m;j}$ given by the 1-periodization process apply to the ℓ-th antiderivative $\psi^{(\ell)}_m$ of the classical Chui-Wang wavelet ψ_m which has the same support. The stability of Chui-Wang wavelets in Sobolev spaces has been studied in [36] and [18]. We use these results to give here a simple proof of the following stability result: if $\frac{1}{2}+\ell-m < s < m+\ell-\frac{1}{2}$, the functions 1 and $2^{j(\ell-s)}\Psi^{(\ell)}_{m;j,k}$, $j \in \mathbb{N}$, $k \in \{0, \ldots, 2^j-1\}$ form a Riesz basis of $H^s_{1-\mathrm{per}}(\mathbb{R})$. For $\ell = m$, the technique used in the proof had to be modified since the functions $\psi^{(m)}_m$ have no more vanishing moments. Some advantages of this construction are that the basis functions are spline functions, they have a short support and are quickly stabilized in the periodization process if ℓ is large; moreover, for $\ell = m$, we prove that those basis functions all vanish at the origin.

The second chapter ends with a study of the dual bases of the Riesz bases we have constructed. We show that for convenient scalar product and order of Sobolev spaces, the dual Riesz basis has also a multiscale frame. We give it explicitly.

Collocation methods using splines are a natural and widely used technique for solving strongly elliptic pseudodifferential equations on closed curves (see [4], [16], [46]). Spline functions are quite natural since they are easy to handle in implementation. Collocation methods are preferred to the Galerkin ones because evaluations at a finite number of points are usually less time consuming than evaluations of scalar products. However, stability, asymptotic convergence, good condition numbers and efficient compression are not so easy to obtain. For smooth boundaries, the convergence of these methods has been proved by D.N. Arnold, J. Saranen and W.L. Wendland ([3], [4], [42]). Several recent papers use these methods in a more general setting (see for example [25], [32], [34]). Numerical investigations show that the numerical computations involved in the resolution of these collocation equations can be ill-conditioned (see for example [30]). The multiscale and wavelets techniques have naturally been applied to these questions since they provide good Riesz bases, allow progressive computations and give good compression schemes (see for example the papers [22], [18]).

In the third chapter, we show how hierarchical periodic spline spaces can be used to approximate solutions for a large class of pseudodifferential equations on boundaries of smooth open subsets of $\mathbb{R}^2$. This large class includes for example equations related to the single and the double layer potential for solving the Dirichlet problem for the Laplace operator. The main point is to check the coercivity condition of Céa lemma. We give a simple proof of the characterization of this coercivity condition, which leads to relations on the meshes and order of the splines easy to handle.

Then we endow the periodic Sobolev space $H^s_{1-\mathrm{per}}(\mathbb{R})$ with a norm equivalent to the natural one which makes the Galerkin system of Céa lemma equivalent to a collocation system for high resolution levels. A first example for the test or the trial bases is to choose one of the Riesz bases constructed in the second chapter. We investigate the asymptotic stability of such systems. An analysis of the proofs of the convergence theorems shows that the condition number of the stiffness matrices is essentially determined by the Riesz bounds of the test and trial bases in some periodic Sobolev spaces.

Finally in the fourth chapter we give some numerical computations of the condition number and of the error. These examples are concerned with the simple and double layer potentials for the Dirichlet problem for the Laplace operator. The first one involves Sobolev spaces of half integer orders and the second one Sobolev space of integer orders. The test and trial bases are chosen among the Riesz bases constructed in the second chapter. We give in this last chapter some useful tools needed to perform the numerical computations.

I would especially like to express all my gratitude to Professor F. Bastin for her constant help during the accomplishment of this work. Her time has never mattered to answer my numerous questions. Her guidance and encouragement have been of inestimable value to me.

I am also thankful to Professor P. Laubin for sharing some of his ideas and results and for his help in handling numerical results.

I would particularly like to express my gratitude to Professor R. Schneider for fruitful discussions and for his invitation in Chemnitz last november.

I also want to thank Professor F.X. Litt for his interest in my work and for a long discussion.

Thanks so much to all the people who helped me in one way or another to accomplish this work, and especially to M. Baiwir, S. Dispa, O. Faure and H. Wendland.

Chapter 1

Construction of biorthogonal wavelets bases in $H^m(\mathbb{R})$

This first chapter is devoted to the construction of biorthogonal wavelets bases in the Sobolev spaces $H^m(\mathbb{R})$ $(m \in \mathbb{N}_0)$. Two constructions are presented. The first one is a Sobolev version of the construction of Cohen, Daubechies and Feauveau (see [15]). The second one generalizes the results of Chui and Wang (see [13]). Both of these constructions are carried out using the tricks and techniques of the papers [9] and [10] of F. Bastin and P. Laubin. Remind that in [9] and [10], the authors build orthonormal wavelets in Sobolev spaces of integer order $H^m(\mathbb{R})$, that are compactly supported and have a arbitrary high regularity. Their work is a Sobolev version of the well known paper [23] of I. Daubechies. It consists in [9] and [10] of a more general construction where the scaling function depends on the level j. Those wavelets can be constructed by a multiresolution frame but, to obtain compactly supported wavelets, it is more convenient to use a filter construction. To this end, F. Bastin and P. Laubin choose a family of filters which are trigonometric polynomials (to obtain compactly supported wavelets) constructed in such a way that they lead to the cancellation of the singularities due to the Sobolev weight.

Before presenting our two constructions, let us remind some definitions and basic properties about Sobolev spaces.

1.1 Sobolev spaces

Definition 1.1 *For every $s \in \mathbb{R}$, we define the Sobolev space of order s as follows*

$$H^s(\mathbb{R}) = \{u \in S'(\mathbb{R}) : (1 + |\cdot|^2)^{\frac{s}{2}}\widehat{u} \in L^2(\mathbb{R})\}$$

Endowed with the norm

$$\|u\|_{H^s(\mathbb{R})} = \sqrt{\frac{1}{2\pi}\int_{\mathbb{R}}(1+\xi^2)^s|\widehat{u}(\xi)|^2 d\xi},$$

the space $H^s(\mathbb{R})$ is a Hilbert space. For $s = m \in \mathbb{N}$, we have the following description

$$H^m(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \mathrm{D}^k f \in L^2(\mathbb{R}),\ \forall k \in \{0, \dots, m\}\}$$

and the expression

$$\|f\|_m = \sqrt{\sum_{k=0}^{m} \|\mathrm{D}^k f\|^2_{L^2(\mathbb{R})}}$$

is an equivalent norm.

Proposition 1.2 *If $s \in]0, +\infty[$, then*

$$H^s(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : (1 + |\cdot|^2)^{\frac{s}{2}} \widehat{f} \in L^2(\mathbb{R})\}$$

and $H^{-s}(\mathbb{R})$ is the dual space of $H^s(\mathbb{R})$, i.e.

$$H^{-s}(\mathbb{R}) = \{u \in D'(\mathbb{R}) : \exists C > 0 \text{ s.t. } |u(\varphi)| \leq C\|\varphi\|_{H^s(\mathbb{R})}, \ \forall \varphi \in D(\mathbb{R})\}$$

∎

Let us also introduce the local Sobolev spaces.

Definition 1.3 *For $s \in \mathbb{R}$, we define the local Sobolev space $H^s_{\mathrm{loc}}(\mathbb{R})$ by*

$$H^s_{\mathrm{loc}}(\mathbb{R}) := \{u \in D'(\mathbb{R}) : \varphi u \in H^s(\mathbb{R}), \ \forall \varphi \in D(\mathbb{R})\}.$$

Proposition 1.4 *If $s \in]0, +\infty[$, then*

$$H^s_{\mathrm{loc}}(\mathbb{R}) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}) : \varphi f \in H^s(\mathbb{R}), \ \forall \varphi \in D(\mathbb{R})\}$$

∎

The following proposition will be useful in the next chapter. It can be directly obtained from Lemma 1.6. This last one is proved in [29].

Proposition 1.5 *Let s be in $]0, +\infty[\setminus \mathbb{N}_0$. If $s = m + \sigma$ with $m \in \mathbb{N}$ and $\sigma \in]0, 1[$, then a function f belongs to $H^s(\mathbb{R})$ if and only if it belongs to $H^m(\mathbb{R})$ and is such that the function*

$$\frac{|\mathrm{D}^m f(x) - \mathrm{D}^m f(y)|^2}{|x - y|^{1+2\sigma}}$$

belongs to $L^1(\mathbb{R}^2)$. The norm

$$\|f\|'_{H^s(\mathbb{R})} = \sqrt{\sum_{\alpha=0}^{m} \|\mathrm{D}^\alpha f\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \frac{|\mathrm{D}^m f(x) - \mathrm{D}^m f(y)|^2}{|x - y|^{1+2\sigma}}}$$

is then equivalent to $\|\cdot\|_{H^s(\mathbb{R})}$ on $H^s(\mathbb{R})$. ∎

Lemma 1.6 *If $s \in]0, 1[$, then for every $f \in H^s(\mathbb{R})$, we have*

$$\frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi = A(s) \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \, \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}}$$

with

$$\frac{1}{A(s)} = \frac{1}{|\xi|^{2s}} \int_{\mathbb{R}} \frac{|e^{ix\xi} - 1|^2}{|x|^{1+2s}} dx, \quad \forall \xi \in \mathbb{R}_0.$$

∎

1.2 Construction of biorthogonal wavelets bases in $H^m(\mathbb{R})$ of Cohen, Daubechies and Feauveau type

Let m be a strictly positive integer.

1.2.1 Riesz basis and Riesz conditions

Let $(H, \langle\cdot\,,\,\cdot\rangle)$ be a Hilbert space and $\|\cdot\|$ the corresponding norm.

Definition 1.7 *A sequence of functions $(f_\ell)_{\ell\in\mathbb{N}_0}$ is called a Riesz basis of H if the following two conditions are satisfied*

- $H = \rangle f_\ell : \ell \in \mathbb{N}_0\langle^-$;
- *there are constants A, B satisfying $0 < A \leq B < +\infty$ such that*

$$A\sqrt{\sum_{(\ell)}|c_\ell|^2} \leq \left\|\sum_{(\ell)} c_\ell f_\ell\right\| \leq B\sqrt{\sum_{(\ell)}|c_\ell|^2}$$

  for every sequence $(c_\ell)_{\ell\in\mathbb{N}_0}$.

The second condition above is called the Riesz condition and the constants A, B the Riesz bounds.

Proposition 1.8 *If $(f_\ell)_{\ell\in\mathbb{N}_0}$ is a Riesz basis of H, the linear map*

$$T : H \to H \quad f \mapsto \sum_{\ell\in\mathbb{N}_0}\langle f\,,\, f_\ell\rangle f_\ell$$

is an isomorphism, the sequence $(T^{-1}f_\ell)_{\ell\in\mathbb{N}_0}$ is a Riesz basis for H and it satisfies

$$\langle T^{-1}f_\ell\,,\, f_{\ell'}\rangle = \delta_{\ell,\ell'}, \quad f = \sum_{\ell\in\mathbb{N}_0}\langle f\,,\, T^{-1}f_\ell\rangle f_\ell = \sum_{\ell\in\mathbb{N}_0}\langle f\,,\, f_\ell\rangle T^{-1}f_\ell.$$

Moreover, if a sequence $(g_\ell)_{\ell\in\mathbb{N}_0}$ of elements of H is such that $\langle f_\ell\,,\, g_k\rangle = \delta_{\ell,k}$, then $g_\ell = T^{-1}f_\ell$ for every $\ell \in \mathbb{N}_0$. ∎

Definition 1.9 *If $(f_\ell)_{\ell\in\mathbb{N}_0}$ is a Riesz basis of H, the sequence $(T^{-1}f_\ell)_{\ell\in\mathbb{N}_0}$ is called the dual of the basis f_ℓ $(\ell \in \mathbb{N}_0)$* (see [12]).

In what follows, we deal with sequences satisfying weaker conditions. This is the reason why we introduce the following definition.

Definition 1.10 *The sequence $(f_\ell)_{\ell\in\mathbb{N}_0}$ satisfies the pre-Riesz condition if there is $B > 0$ such that*

$$\left\|\sum_{(\ell)} c_\ell f_\ell\right\| \leq B\sqrt{\sum_{(\ell)}|c_\ell|^2}$$

for every sequence $(c_\ell)_{\ell\in\mathbb{N}_0}$. We say that B is a bound for the family f_ℓ $(\ell \in \mathbb{N}_0)$.

In this case, for every $f \in H$, the series $\sum_{\ell \in \mathbb{N}_0} |\langle f \,,\, f_\ell \rangle|^2$ converges and

$$\sqrt{\sum_{\ell \in \mathbb{N}_0} |\langle f \,,\, f_\ell \rangle|^2} \leq B\|f\|.$$

Indeed, we have

$$\sum_{(\ell)} |\langle f \,,\, f_\ell \rangle|^2 = \langle f \,,\, \sum_{(\ell)} \langle f \,,\, f_\ell \rangle f_\ell \rangle \leq \|f\| \; \|\sum_{(\ell)} \langle f \,,\, f_\ell \rangle f_\ell\| \leq B\|f\| \; \sqrt{\sum_{(\ell)} |\langle f \,,\, f_\ell \rangle|^2}.$$

Moreover, if $(g_\ell)_{\ell \in \mathbb{N}_0}$ is another sequence satisfying the pre-Riesz condition with bound B', the operator

$$P \;:\; H \to H \quad f \mapsto \sum_{\ell \in \mathbb{N}_0} \langle f \,,\, g_\ell \rangle f_\ell$$

is well defined, linear, continuous and satisfies

$$\begin{aligned}
\|P\| &= \sup_{\|g\|, \|f\| \leq 1} |\langle Pf \,,\, g \rangle| = \sup_{\|g\|, \|f\| \leq 1} |\sum_{\ell \in \mathbb{N}_0} \langle f_\ell \,,\, g \rangle \langle f \,,\, g_\ell \rangle| \\
&\leq \sup_{\|g\|, \|f\| \leq 1} \sqrt{\sum_{\ell \in \mathbb{N}_0} |\langle f \,,\, g_\ell \rangle|^2} \sqrt{\sum_{\ell \in \mathbb{N}_0} |\langle g \,,\, f_\ell \rangle|^2} \\
&\leq BB'.
\end{aligned}$$

If in addition the families f_ℓ ($\ell \in \mathbb{N}_0$) and g_ℓ ($\ell \in \mathbb{N}_0$) satisfy the Riesz condition, they are Riesz bases for $V = \rangle f_\ell \;:\; \ell \in \mathbb{N}_0 \langle^{-}$, $\widetilde{V} = \rangle g_\ell \;:\; \ell \in \mathbb{N}_0 \langle^{-}$ respectively; and if $\langle f_\ell \,,\, g_k \rangle = \delta_{\ell,k}$ for every $\ell, k \in \mathbb{N}_0$, the map P is a projection onto V; the maps P and $I - P$ are in fact the projections associated to the direct sum $H = V \oplus \widetilde{V}^{\perp}$. In particular, if $g_\ell \in V$ for every ℓ, then it is the dual of the family f_ℓ ($\ell \in \mathbb{N}_0$) and P is the orthogonal projection onto V.

1.2.2 Multiresolution analysis point of view in Sobolev spaces

Following ([12]), we give the definition below.

Definition 1.11 *We call biorthogonal wavelets in $H^m(\mathbb{R})$ two families of functions $\psi_{j,k}$ ($j, k \in \mathbb{Z}$) and $\widetilde{\psi}_{j,k}$ ($j, k \in \mathbb{Z}$) such that*

- $\psi_{j,k}(x) := 2^{j/2}\psi^{(j)}(2^j x - k)$ ($j, k \in \mathbb{Z}$) *is a Riesz basis of* $H^m(\mathbb{R})$;
- $\widetilde{\psi}_{j,k}(x) := 2^{j/2}\widetilde{\psi}^{(j)}(2^j x - k)$ ($j, k \in \mathbb{Z}$) *is a Riesz basis of* $H^m(\mathbb{R})$;
- $\langle \psi_{j,k} \,,\, \widetilde{\psi}_{j',k'} \rangle_{H^m(\mathbb{R})} = \delta_{j,j'}\delta_{k,k'} \; \forall j, j', k, k' \in \mathbb{Z}$.

We say that the functions $\psi^{(j)}, \widetilde{\psi}^{(j)}$ are dual to each other. We recall that the index j means that we work with one mother wavelet for each scale; this comes from the form of the norm of the Sobolev space. To simplify the notations, we will write $\widehat{\psi}^{(j)}$ (resp. $\widehat{\widetilde{\psi}}^{(j)}$) instead of $\widehat{\psi^{(j)}}$ (resp. $\widehat{\widetilde{\psi}^{(j)}}$).

To obtain such bases, we want to adapt the construction coming from the general theory of wavelets. We first give the following result. It is a consequence of the pre-Riesz (resp. Riesz) condition; it is the generalization of what happens in the orthonormal case (see [35],[9]).

Proposition 1.12 *Let $\varphi \in H^m(\mathbb{R})$ and $j \in \mathbb{Z}$. The following conditions are equivalent.*

1. *The family $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$ $(k \in \mathbb{Z})$ satisfies the pre-Riesz condition with bound B_j (resp. the Riesz condition with bounds A_j, B_j).*

2. *We have*

$$(\text{resp. } A_j^2 \leq \quad)\sum_{k\in\mathbb{Z}}(1+2^{2j}(\xi+2k\pi)^2)^m|\widehat{\varphi}(\xi+2k\pi)|^2 \;\leq\; B_j^2.$$

PROOF. See [6]. ∎

In case we have a good behavior for the different scales, i.e. good behavior in terms of j, we get the following results.

Proposition 1.13 *For every $j \in \mathbb{Z}$, let $\varphi^{(j)}$ and $\widetilde{\varphi}^{(j)}$ be elements of $H^m(\mathbb{R})$. Assume that the family $\varphi_{j,k}(x) = 2^{j/2}\varphi^{(j)}(2^j x - k)$ $(k \in \mathbb{Z})$ (resp. $\widetilde{\varphi}_{j,k}(x) = 2^{j/2}\widetilde{\varphi}^{(j)}(2^j x - k)$ $(k \in \mathbb{Z})$) satisfy the pre-Riesz condition with bound B (resp. $\widetilde{B}$) independent of j. We denote by P_j the map*

$$P_j \;: H^m(\mathbb{R}) \to H^m(\mathbb{R}) \quad f \;\mapsto \sum_{k\in\mathbb{Z}}\langle f\,,\,\widetilde{\varphi}_{j,k}\rangle_{H^m(\mathbb{R})}\varphi_{j,k}.$$

a) If

$$\lim_{j\to+\infty}\overline{\widehat{\varphi^{(j)}}}(2^{-j}\xi)\,\widehat{\widetilde{\varphi}}^{(j)}(2^{-j}\xi) = (1+\xi^2)^{-m} \quad \text{a.e.}$$

then, for every $f, g \in H^m(\mathbb{R})$,

$$\lim_{j\to+\infty}\langle P_j f\,,\,g\rangle_{H^m(\mathbb{R})} = \langle f\,,\,g\rangle_{H^m(\mathbb{R})}.$$

b) If there are $A, \alpha > 0$ such that

$$\int_{\mathbb{R}}(1+|\xi|)^\alpha|\widehat{\widetilde{\varphi}}^{(j)}(\xi)|^2\,d\xi \;\leq\; A, \quad \forall j \leq 0,$$

then, for every $f \in H^m(\mathbb{R})$,

$$\lim_{j\to-\infty}\|P_j f\|_{H^m(\mathbb{R})} = 0.$$

PROOF. See [6]. ∎

Corollary 1.14 *Under the hypothesis of the previous proposition, we have*

$$f = \sum_{j\in\mathbb{Z}} (P_{j+1} - P_j)(f) \quad \textit{weakly.}$$

PROOF. It suffices to write $P_J = \sum_{j=-J-1}^{J-1} (P_{j+1} - P_j) + P_{-J-1}$ for all $J \in \mathbb{N}$, and to apply the previous result. ∎

For every $j \in \mathbb{Z}$, we define V_j (resp. $\widetilde{V}_j$) as the closed linear hull of the sequence $\varphi_{j,k}$ $(k \in \mathbb{Z})$ (resp. $\widetilde{\varphi}_{j,k}$ $(k \in \mathbb{Z})$). In the context of wavelets, we will have $V_j \subset V_{j+1}$ and $\widetilde{V}_j \subset \widetilde{V}_{j+1}$ for every $j \in \mathbb{Z}$. In the best situation, for every $j \in \mathbb{Z}$, the families $\varphi_{j,k}$ $(k \in \mathbb{Z})$ and $\widetilde{\varphi}_{j,k}$ $(k \in \mathbb{Z})$ are Riesz bases for V_j and $\widetilde{V}_j$ respectively and are such that $\langle \varphi_{j,k}, \widetilde{\varphi}_{j,k'} \rangle = \delta_{k,k'}$. Hence the maps $P_{j+1} - P_j$ are projections onto $W_j := V_{j+1} \cap \widetilde{V}_j^{\perp}$ and we get good candidates for biorthogonal wavelets constructing Riesz bases for W_j $(j \in \mathbb{Z})$.

In the general situation, it may happen that the family $\varphi_{j,k}$ $(k \in \mathbb{Z})$ or $\widetilde{\varphi}_{j,k}$ $(k \in \mathbb{Z})$ is not a Riesz basis. We have weaker assumptions on them, coming from information on filters. This was already the case in [15]; here we present a generalization of these results : we work in Sobolev spaces and have in mind to use the filters introduced in [9] to get compact support.

1.2.3 General result

Here follows a Sobolev version of Theorem 3.8 of ([15]). For every $j \in \mathbb{Z}$, let $m_0^{(j)}(\xi)$ and $\widetilde{m_0}^{(j)}(\xi)$ be trigonometric polynomials such that

$$\overline{m_0^{(j)}}(\xi)\widetilde{m_0}^{(j)}(\xi) + \overline{m_0^{(j)}}(\xi+\pi)\widetilde{m_0}^{(j)}(\xi+\pi) = 1. \tag{1.1}$$

Assume also that there is $C > 0$ such that

$$\sup_{j\in\mathbb{Z}} \sup_{\xi\in\mathbb{R}} |m_0^{(j)}(\xi)| \le C, \quad \sup_{j\in\mathbb{Z}} \sup_{\xi\in\mathbb{R}} |\widetilde{m_0}^{(j)}(\xi)| \le C. \tag{1.2}$$

Let also $\varphi^{(j)}, \widetilde{\varphi}^{(j)}$ be elements of $H^m(\mathbb{R})$ such that
- the following scaling relations hold for every j :

$$\widehat{\varphi}^{(j)}(2\xi) = m_0^{(j+1)}(\xi)\widehat{\varphi}^{(j+1)}(\xi), \;\; \widehat{\widetilde{\varphi}}^{(j)}(2\xi) = \widetilde{m_0}^{(j+1)}(\xi)\widehat{\widetilde{\varphi}}^{(j+1)}(\xi);$$

- the Fourier transforms satisfy

$$\left.\begin{array}{l} |\widehat{\varphi}^{(j)}(\xi)| \\ |\widehat{\widetilde{\varphi}}^{(j)}(\xi)| \end{array}\right\} \le \frac{A}{(1+2^{2j}\xi^2)^{m/2}} \frac{1}{(1+|\xi|)^{\frac{1}{2}+\varepsilon}},$$

for some $\varepsilon, A > 0$;
- $\lim_{j\to+\infty} \overline{\widehat{\varphi}^{(j)}}(2^{-j}\xi)\widehat{\widetilde{\varphi}}^{(j)}(2^{-j}\xi) = (1+\xi^2)^{-m}$.

We define $\widehat{\psi}^{(j)}$ and $\widehat{\widetilde{\psi}}^{(j)}$ as follows

$$\begin{aligned}\widehat{\psi}^{(j)}(2\xi) &:= -e^{-i\xi}\overline{\widetilde{m_0}^{(j+1)}}(\xi+\pi)\widehat{\varphi}^{(j+1)}(\xi) = m_1^{(j+1)}(\xi)\widehat{\varphi}^{(j+1)}(\xi)\\ \widehat{\widetilde{\psi}}^{(j)}(2\xi) &:= -e^{-i\xi}\overline{m_0^{(j+1)}}(\xi+\pi)\widehat{\widetilde{\varphi}}^{(j+1)}(\xi) = \widetilde{m_1}^{(j+1)}(\xi)\widehat{\widetilde{\varphi}}^{(j+1)}(\xi)\end{aligned}$$

and

$$\psi_{j,k}(x) := 2^{\frac{j}{2}}\psi^{(j)}(2^j x - k), \quad \widetilde{\psi}_{j,k}(x) := 2^{\frac{j}{2}}\widetilde{\psi}^{(j)}(2^j x - k), \quad j,k \in \mathbb{Z}.$$

Remark 1.15 *For a fixed index j and in case we deal with trigonometric polynomials, the condition of exact reconstruction*

$$\begin{cases} \widetilde{m_0}^{(j)}(\xi)\,\overline{m_0^{(j)}}(\xi) + \widetilde{m_1}^{(j)}(\xi)\,\overline{m_1^{(j)}}(\xi) & = 1\\ \widetilde{m_0}^{(j)}(\xi)\,\overline{m_0^{(j)}}(\xi+\pi) + \widetilde{m_1}^{(j)}(\xi)\,\overline{m_1^{(j)}}(\xi+\pi) & = 0\end{cases}$$

is equivalent to

$$\begin{cases} \overline{m_0^{(j)}}(\xi)\widetilde{m_0}^{(j)}(\xi) + \overline{m_0^{(j)}}(\xi+\pi)\widetilde{m_0}^{(j)}(\xi+\pi) = 1\\ \exists c \in \mathbb{C}\setminus\{0\}, k \in 2\mathbb{Z}+1 : m_1^{(j)}(\xi) = \overline{c}^{-1}e^{ik\xi}\overline{\widetilde{m_0}^{(j)}}(\xi+\pi), \widetilde{m_1}^{(j)}(\xi) = ce^{ik\xi}\overline{m_0^{(j)}}(\xi+\pi).\end{cases}$$

See for example [15].

First, we check that the hypothesis of Proposition 1.13 are satisfied. The additional ones in a) and b) are clearly satisfied in view of the assumptions, as well as the uniform boundedness of the Riesz bounds as it is shown below.

Proposition 1.16 *For every $j \in \mathbb{Z}$, the four families $\{\varphi_{j,k} : k \in \mathbb{Z}\}$, $\{\widetilde{\varphi}_{j,k} : k \in \mathbb{Z}\}$, $\{\psi_{j,k} : k \in \mathbb{Z}\}$, $\{\widetilde{\psi}_{j,k} : k \in \mathbb{Z}\}$ satisfy the pre-Riesz condition. Moreover, the bounds are independent of j.*

PROOF. See [6]. ∎

For every $j \in \mathbb{Z}$, we define then the linear and continuous operators

$$\begin{aligned} P_j : H^m(\mathbb{R}) \to H^m(\mathbb{R}) \quad & f \mapsto \sum_{k\in\mathbb{Z}}\langle f, \widetilde{\varphi}_{j,k}\rangle_{H^m(\mathbb{R})}\varphi_{j,k};\\ \widetilde{P}_j : H^m(\mathbb{R}) \to H^m(\mathbb{R}) \quad & f \mapsto \sum_{k\in\mathbb{Z}}\langle f, \varphi_{j,k}\rangle_{H^m(\mathbb{R})}\widetilde{\varphi}_{j,k}\end{aligned}$$

and

$$\begin{aligned} Q_j : H^m(\mathbb{R}) \to H^m(\mathbb{R}) \quad & f \mapsto \sum_{k\in\mathbb{Z}}\langle f, \widetilde{\psi}_{j,k}\rangle_{H^m(\mathbb{R})}\psi_{j,k};\\ \widetilde{Q}_j : H^m(\mathbb{R}) \to H^m(\mathbb{R}) \quad & f \mapsto \sum_{k\in\mathbb{Z}}\langle f, \psi_{j,k}\rangle_{H^m(\mathbb{R})}\widetilde{\psi}_{j,k}.\end{aligned}$$

Corollary 1.17 *For every $j \in \mathbb{Z}$, we have*

$$P_{j+1} - P_j = Q_j \quad \text{et} \quad \widetilde{P}_{j+1} - \widetilde{P}_j = \widetilde{Q}_j.$$

PROOF. First of all, let us introduce some useful notations. For every integer j, we denote by $\frac{h_n^{(j)}}{\sqrt{2}}$ (resp. $\frac{\widetilde{h}_n^{(j)}}{\sqrt{2}}$) the coefficients of the trigonometric polynomial $m_0^{(j)}$ (resp. $\widetilde{m_0}^{(j)}$), i.e.

$$m_0^{(j)}(\xi) = \frac{1}{\sqrt{2}} \sum_{(n\in\mathbb{Z})} h_n^{(j)} e^{-in\xi}, \quad \widetilde{m_0}^{(j)}(\xi) = \frac{1}{\sqrt{2}} \sum_{(n\in\mathbb{Z})} \widetilde{h}_n^{(j)} e^{-in\xi}.$$

The coefficients $g_n^{(j)}$ and $\widetilde{g}_n^{(j)}$, $j, n \in \mathbb{Z}$, are defined by

$$g_n^{(j)} := (-1)^n \, \overline{h_{-n+1}^{(j)}}, \quad \widetilde{g}_n^{(j)} := (-1)^n \, \overline{\widetilde{h}_{-n+1}^{(j)}}.$$

So, let $j \in \mathbb{Z}$ and $f_1 \in H^m(\mathbb{R})$. One has merely to prove that we have

$$\sum_{k\in\mathbb{Z}} \langle f_1 \,,\, \widetilde{\varphi}_{j,k} \rangle_{H^m(\mathbb{R})} \langle \varphi_{j,k} \,,\, f_2 \rangle_{H^m(\mathbb{R})} + \sum_{k\in\mathbb{Z}} \langle f_1 \,,\, \widetilde{\psi}_{j,k} \rangle_{H^m(\mathbb{R})} \langle \psi_{j,k} \,,\, f_2 \rangle_{H^m(\mathbb{R})}$$
$$= \sum_{k\in\mathbb{Z}} \langle f_1 \,,\, \widetilde{\varphi}_{j+1,k} \rangle_{H^m(\mathbb{R})} \langle \varphi_{j+1,k} \,,\, f_2 \rangle_{H^m(\mathbb{R})}$$

for any $f_2 \in H^m(\mathbb{R})$. Let $f_2 \in H^m(\mathbb{R})$. We get

$$\sum_{k\in\mathbb{Z}} \Big(\langle f_1 \,,\, \widetilde{\varphi}_{j,k} \rangle_{H^m(\mathbb{R})} \langle \varphi_{j,k} \,,\, f_2 \rangle_{H^m(\mathbb{R})} + \langle f_1 \,,\, \widetilde{\psi}_{j,k} \rangle_{H^m(\mathbb{R})} \langle \psi_{j,k} \,,\, f_2 \rangle_{H^m(\mathbb{R})} \Big)$$
$$= \sum_{k\in\mathbb{Z}} \sum_{(n\in\mathbb{Z})} \sum_{(\ell\in\mathbb{Z})} \Big(\overline{\widetilde{h}_n^{(j+1)}} \, h_\ell^{(j+1)} + \overline{\widetilde{g}_n^{(j+1)}} \, g_\ell^{(j+1)} \Big) \langle f_1 \,,\, \widetilde{\varphi}_{j+1,2k+n} \rangle_{H^m(\mathbb{R})} \langle \varphi_{j+1,2k+\ell} \,,\, f_2 \rangle_{H^m(\mathbb{R})}$$
$$= \sum_{n\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} \langle f_1 \,,\, \widetilde{\varphi}_{j+1,n} \rangle_{H^m(\mathbb{R})} \langle \varphi_{j+1,\ell} \,,\, f_2 \rangle_{H^m(\mathbb{R})} \sum_{k\in\mathbb{Z}} \Big(\overline{\widetilde{h}_{n-2k}^{(j+1)}} \, h_{\ell-2k}^{(j+1)} + \overline{\widetilde{g}_{n-2k}^{(j+1)}} \, g_{\ell-2k}^{(j+1)} \Big)$$
$$= \sum_{n\in\mathbb{Z}} \langle f_1 \,,\, \widetilde{\varphi}_{j+1,n} \rangle_{H^m(\mathbb{R})} \langle \varphi_{j+1,n} \,,\, f_2 \rangle_{H^m(\mathbb{R})}.$$

∎

Proposition 1.18 *For every $f, g \in H^m(\mathbb{R})$, we have*

$$\lim_{j\to+\infty} \langle P_j f \,,\, g \rangle_{H^m(\mathbb{R})} = \langle f \,,\, g \rangle_{H^m(\mathbb{R})}, \quad \lim_{j\to+\infty} \| P_{-j}(f) \|_{H^m(\mathbb{R})} = 0;$$
$$\lim_{j\to+\infty} \langle \widetilde{P}_j f \,,\, g \rangle_{H^m(\mathbb{R})} = \langle f \,,\, g \rangle_{H^m(\mathbb{R})}, \quad \lim_{j\to+\infty} \| \widetilde{P}_{-j}(f) \|_{H^m(\mathbb{R})} = 0.$$

PROOF. It is a consequence of Propositions 1.13 and 1.16. ∎

Corollary 1.19 *For every $f \in H^m(\mathbb{R})$, we have, in the weak sense,*

$$f = \lim_{J\to+\infty} \sum_{j=-J}^{J} \sum_{k\in\mathbb{Z}} \langle f \,,\, \widetilde{\psi}_{j,k} \rangle_{H^m(\mathbb{R})} \psi_{j,k} = \lim_{J\to+\infty} \sum_{j=-J}^{J} \sum_{k\in\mathbb{Z}} \langle f \,,\, \psi_{j,k} \rangle_{H^m(\mathbb{R})} \widetilde{\psi}_{j,k}.$$

PROOF. This is obtained from $P_{j+1} - P_j = Q_j$ and $f = \sum_{j\in\mathbb{Z}}(P_{j+1} - P_j)(f)$ weakly (and from the similar formulae with the "$\sim$" marks). ∎

Proposition 1.22 states that under an assumption on the filters, the families $\{\psi_{j,k} : \ j,k \in \mathbb{Z}\}$ and $\{\widetilde{\psi}_{j,k} : \ j,k \in \mathbb{Z}\}$ are frames for $H^m(\mathbb{R})$. It uses some techniques of the first part of Lemma 3.4, p.501, of [15] in the context of Sobolev spaces.

Let us first recall the definition of a frame of a Hilbert space $(H, \|\cdot\|)$ as well as the relationship between a frame and a Riesz basis. For more information, we refer to [12].

Definition 1.20 *A sequence of functions $(f_\ell)_{\ell\in\mathbb{N}_0}$ of H is called a frame of $(H, \|.\|)$ if there are $A, B > 0$ (these numbers are called frame bounds) such that*

$$A^2\|f\|^2 \leq \sum_{\ell=1}^{+\infty} |\langle f\,,\, f_\ell\rangle|^2 \leq B^2\|f\|^2$$

for every $f \in H$.

A Riesz basis is always a frame, with the same bounds. The converse holds in case the f_ℓ, $\ell \in \mathbb{N}_0$, are ℓ^2-linearly independent; more precisely, we have the following proposition.

Proposition 1.21 *T.f.a.e.*

a) *The sequence $(f_\ell)_{\ell\in\mathbb{N}_0}$ is a Riesz basis for $(H, \|.\|)$.*

b) *The sequence $(f_\ell)_{\ell\in\mathbb{N}_0}$ is a frame for $(H, \|.\|)$ and satisfies the following property*

$$\Big((c_\ell)_{\ell\in\mathbb{N}_0} \in \ell^2 \quad \text{and} \quad \sum_{\ell=1}^{+\infty} c_\ell f_\ell = 0 \text{ in } H\Big) \Rightarrow \Big(c_\ell = 0,\ \forall \ell \in \mathbb{N}_0\Big).$$

∎

Proposition 1.22 *Assume there is $E > 0$ such that*

$$\sup_{j\in\mathbb{Z}} |m_0^{(j)}(\xi + \pi)| \leq E|\xi|, \quad \sup_{j\in\mathbb{Z}} |\widetilde{m_0}^{(j)}(\xi + \pi)| \leq E|\xi|.$$

Then there are $c_1, c_2 > 0$ such that

$$c_1\|f\|^2_{H^m(\mathbb{R})} \leq \left\{ \begin{array}{l} \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f\,,\, \psi_{j,k}\rangle_{H^m(\mathbb{R})}|^2 \\ \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f\,,\, \widetilde{\psi}_{j,k}\rangle_{H^m(\mathbb{R})}|^2 \end{array} \right. \leq c_2\|f\|^2_{H^m(\mathbb{R})}$$

for every $f \in H^m(\mathbb{R})$.

PROOF. See [6]. ∎

Corollary 1.23 *In Corollary 1.19, the convergence holds in $H^m(\mathbb{R})$.* ∎

1.2.4 Construction of particular filters

Our aim is to construct real biorthogonal wavelets in the Sobolev space $H^m(\mathbb{R})$ with compact support, regularity and symmetry axis. A natural way to get them (see [15], [9], [10], [11] and the previous general result) is to get special families of filters which are taken to be $2\pi-$periodic trigonometric polynomials $m_0^{(j)}, \widetilde{m_0}^{(j)}$ $(j \in \mathbb{Z})$ with real coefficients such that

$$m_0^{(j)}(\xi)\,\overline{\widetilde{m_0}^{(j)}}(\xi) \;+\; m_0^{(j)}(\xi+\pi)\,\overline{\widetilde{m_0}^{(j)}}(\xi+\pi) = 1, \quad \forall \xi \in \mathbb{R},\ \forall j \in \mathbb{Z}. \tag{1.3}$$

These functions are called filters. The scaling functions $\varphi^{(j)}$ and $\widetilde{\varphi}^{(j)}$ are then defined from the filters as follows

$$\begin{aligned}\widehat{\varphi}^{(j)}(\xi) &:= \frac{1}{(1+i2^j\xi)^m}\prod_{p=1}^{+\infty} m_0^{(j+p)}(2^{-p}\xi)\\ \widehat{\widetilde{\varphi}}^{(j)}(\xi) &:= \frac{1}{(1+i2^j\xi)^m}\prod_{p=1}^{+\infty} \widetilde{m_0}^{(j+p)}(2^{-p}\xi)\end{aligned}$$

and the wavelets are defined by

$$\widehat{\psi}^{(j)}(2\xi) := -e^{-i\xi}\,\overline{\widetilde{m_0}^{(j+1)}}(\xi+\pi)\widehat{\varphi}^{(j+1)}(\xi), \quad \widehat{\widetilde{\psi}}^{(j)}(2\xi) := -e^{-i\xi}\,\overline{m_0^{(j+1)}}(\xi+\pi)\widehat{\widetilde{\varphi}}^{(j+1)}(\xi).$$

To obtain compact support and regularity for the functions $\psi^{(j)}, \widetilde{\psi}^{(j)}$, as well as a symmetry axis, we use filters containing "good" factors (see [23], [9], [10], [11]). Here we will present some convenient filters. To introduce them in an easy way, we will first of all present some definitions and notations.

Let N and $\widetilde{N}$ be two natural numbers different from 0 such that their sum $N+\widetilde{N}$ is an even number. Then, we define μ by

$$\mu = \begin{cases} 1 & \text{if } N \text{ is odd} \\ 0 & \text{if } N \text{ is even.}\end{cases}$$

For every $j \in \mathbb{N}$, the function $M^{(j)}$ is defined as follows

$$M^{(j)}(\xi) := e^{i\frac{\xi}{2}}\frac{\cos(\frac{\xi+i2^{-j}}{2})}{\cosh(2^{-j-1})}$$

It will be used to cancel the singularity introduced by the Sobolev norm. For every $j \in \mathbb{Z}$, we also define the polynomial $R^{(j)}_{N,\widetilde{N},m}$ by

$$R^{(j)}_{N,\widetilde{N},m}(\xi) := \left(\frac{x_j}{2x_j-1}\right)^m P^{(j)}_{\frac{N+\widetilde{N}}{2},m}\left(\sin^2(\frac{\xi}{2})\right)$$

where (see [10]) $P^{(j)}_{M,m}(y)$ are the polynomials with real coefficients of degree at most $m+M-1$ defined as the solution $Q^{(j)}_{M,m}(y)$ of lowest degree of the equations

$$(x_j-y)^m(1-y)^M Q^{(j)}_{M,m}(y) + (x_j-1+y)^m y^M Q^{(j)}_{M,m}(1-y) = (2x_j-1)^m \tag{1.4}$$

where $x_j := \cosh^2(2^{-j-1})$.

Then, for m fixed in $\mathbb{N}_0$, the chosen filters are the following

$$m_0^{(j),(N)}(\xi) := (M^{(j)}(\xi))^m \cos^N(\frac{\xi}{2})e^{-i\mu\xi/2}$$
$$\widetilde{m}_0^{(j),(N,\tilde{N})}(\xi) := (M^{(j)}(\xi))^m \cos^{\tilde{N}}(\frac{\xi}{2})e^{-i\mu\xi/2}R^{(j)}_{N,\tilde{N},m}(\xi)$$

Now, let us recall some properties of the polynomials $P^{(j)}_{M,m}$ ($j \in \mathbb{Z}$, $M \in \mathbb{N}$, $m \in \mathbb{N}_0$).

Proposition 1.24 *For any $j \in \mathbb{Z}$, $M \in \mathbb{N}$, $m \in \mathbb{N}_0$, we have*

1. $0 \leq P^{(j)}_{M,m}(y) \leq \begin{cases} 2^{M+m-1} & \text{for } y \in [0, \frac{1}{2}] \\ 2^{2(M+m-1)} & \text{for } y \in [\frac{1}{2}, 1]; \end{cases}$

2. $P^{(j)}_{M,m}(0) = \left(\frac{2x_j - 1}{x_j}\right)^m$;

3. $P^{(j)}_{M,1}(y) = 2\sum_{k=0}^{M-1} C^k_{M+k-1}y^k + (2y-1)\sum_{k=0}^{M-1} y^k x_j^{-k-1} \sum_{\ell=0}^{k} C^{\ell}_{M+\ell-1}x_j^{\ell}$.

PROOF.

1. We have (see [9], [10])

$$P^{(j)}_{M,m}(y) \leq 2^m P_M(y) \leq 2^{M+m-1}$$

for $y \in [0, \frac{1}{2}]$, and

$$P^{(j)}_{M,m}(y) \leq P_{M+m}(y) \leq 2^{2(M+m-1)}$$

for $y \in [\frac{1}{2}, 1]$. For $N \in \mathbb{N}_0$, we have denoted by P_N the Daubechies polynomial of order N, i.e.

$$P_N(y) = \sum_{k=0}^{N-1} C^k_{N+k-1}y^k.$$

2. This is direct from (1.4).

3. See [9].

The proposition is proved. ∎

Using these properties and techniques of ([9]), we check that these filters and the corresponding scaling functions satisfy the properties introduced in subsection 1.2.3 under some hypothesis on N and $\tilde{N}$.

For m fixed in $\mathbb{N}_0$, we consider the functions $\varphi^{(j),(N)}$, $\widetilde{\varphi}^{(j),(N,\tilde{N})}$, $\psi^{(j),(N,\tilde{N})}$, $\widetilde{\psi}^{(j),(N,\tilde{N})}$ defined from the filters $m_0^{(j),(N)}$ and $\widetilde{m}_0^{(j),(N,\tilde{N})}$ by infinite product as in the beginning of this section. We obtain

$$\widehat{\varphi}^{(j),(N)}(\xi) = \left(\frac{-i}{1+2^{2j}\xi^2}\right)^m e^{i(m-\mu)\frac{\xi}{2}} \frac{\sin^m\left(\frac{\xi+i2^{-j}}{2}\right)}{\sinh^m(2^{-j-1})} \left(\frac{\sin(\frac{\xi}{2})}{\frac{\xi}{2}}\right)^N$$

$$\widehat{\widetilde{\varphi}}^{(j),(N,\tilde{N})}(\xi) = \left(\frac{-i2^{j+1}}{1+2^{2j}\xi^2}\right)^m e^{i(m-\mu)\frac{\xi}{2}} \frac{\sin^m\left(\frac{\xi+i2^{-j}}{2}\right)}{\cosh^m(2^{-j-1})} \left(\frac{\sin(\frac{\xi}{2})}{\frac{\xi}{2}}\right)^{\tilde{N}} \prod_{p=1}^{+\infty} P^{(j+p)}_{\frac{N+\tilde{N}}{2},m}\left(\sin^2(2^{-p-1}\xi)\right)$$

and

$$\widehat{\psi}^{(j),(N,\tilde{N})}(2\xi)$$
$$= \frac{(-1)^{\tilde{N}+1} i^\mu e^{-i\xi}}{(1+2^{2j+2}\xi^2)^m} \left(\frac{(\cosh(2^{-j-1})-\cos(\xi))\cosh(2^{-j-2})}{2\cosh(2^{-j-1})\sinh(2^{-j-2})}\right)^m \frac{\sin^{N+\tilde{N}}(\frac{\xi}{2})}{(\frac{\xi}{2})^N} P^{(j+1)}_{\frac{N+\tilde{N}}{2},m}\left(\cos^2(\frac{\xi}{2})\right)$$

$$\widehat{\widetilde{\psi}}^{(j),(N,\tilde{N})}(2\xi)$$
$$= \frac{(-1)^{N+1} i^\mu 2^{(j+1)m} e^{-i\xi}}{(1+2^{2j+2}\xi^2)^m} \left(\frac{\cosh(2^{-j-1})-\cos(\xi)}{\cosh^2(2^{-j-2})}\right)^m \frac{\sin^{N+\tilde{N}}(\frac{\xi}{2})}{(\frac{\xi}{2})^{\tilde{N}}} \prod_{p=1}^{+\infty} P^{(j+1+p)}_{\frac{N+\tilde{N}}{2},m}\left(\sin^2(2^{-p-1}\xi)\right).$$

Theorem 1.25 *a) For these filters, we have*

$$\cos^2(\frac{\xi}{2}) \le |M^{(j)}(\xi)|^2 \le 1 = M^{(j)}(0), \quad \forall \xi \in \mathbb{R},\ \forall j \in \mathbb{Z}$$

hence

$$|m_0^{(j),(N)}(\xi)| \le 1, \quad |\widetilde{m_0}^{(j),(N,\tilde{N})}(\xi)| \le 2^{N+\tilde{N}+2m-2}, \quad \forall \xi \in \mathbb{R},\ \forall j \in \mathbb{Z}$$

and there is $E > 0$ such that

$$\left\{ \begin{array}{l} \sup_{j\in\mathbb{Z}} |m_0^{(j),(N)}(\xi+\pi)| \\ \sup_{j\in\mathbb{Z}} |\widetilde{m_0}^{(j),(N,\tilde{N})}(\xi+\pi)| \end{array} \right. \le E|\xi|.$$

b) For the corresponding scaling functions, there is $c > 0$ such that

$$\left|\widehat{\varphi}^{(j),(N)}(\xi)\right| \le \frac{c}{(1+2^{2j}\xi^2)^{\frac{m}{2}}(1+|\xi|)^N}, \quad \forall \xi \in \mathbb{R},\ \forall j \in \mathbb{Z}$$

and, for every $\varepsilon > 0$, there is $c_\varepsilon > 0$ such that

$$\left|\widehat{\widetilde{\varphi}}^{(j),(N,\tilde{N})}(\xi)\right| \le \frac{c_\varepsilon}{(1+2^{2j}\xi^2)^{\frac{m}{2}}(1+|\xi|)^{\tilde{N}-(\tilde{N}+N+2m-2)\frac{\ln(3)}{2\ln(2)}-\varepsilon}}, \quad \forall \xi \in \mathbb{R},\ \forall j \in \mathbb{Z}.$$

c) If the condition

$$\tilde{N} - (\tilde{N}+N+2m-2)\frac{\ln(3)}{2\ln(2)} > \frac{1}{2}$$

is satisfied, then the functions $\psi^{(j),(N,\tilde N)}$ $(j \in \mathbb{Z})$ *and* $\widetilde{\psi}^{(j),(N,\tilde N)}$ $(j \in \mathbb{Z})$ *are real biorthogonal regular wavelets with compact support independent of* j *and symmetry axis. More precisely, they satisfy*

$$\psi^{(j),(N,\tilde N)}(-x+\frac{1}{2}) = (-1)^{\tilde N}\psi^{(j),(N,\tilde N)}(x+\frac{1}{2})$$
$$\widetilde{\psi}^{(j),(N,\tilde N)}(-x+\frac{1}{2}) = (-1)^{N}\widetilde{\psi}^{(j),(N,\tilde N)}(x+\frac{1}{2}).$$

PROOF. a) We have

$$|M^{(j)}(\xi)|^2 = \frac{\cosh^2(2^{-j-1}) - \sin^2(\xi/2)}{\cosh^2(2^{-j-1})} = \frac{\cosh(2^{-j}) + \cos(\xi)}{\cosh(2^{-j}) + 1}$$

hence

$$\cos^2(\frac{\xi}{2}) \leq |M^{(j)}(\xi)|^2 \leq 1.$$

The conclusion follows then directly, recalling also the uniform estimation on the polynomials $P^{(j)}_{\frac{N+\tilde N}{2},m}$.

b) In what concerns the first estimation, we have for $j \geq 0$

$$\begin{aligned} |\widehat{\varphi}^{(j),(N)}(\xi)| &\leq \frac{C_1}{(1+2^{2j}\xi^2)^m}\frac{(\xi^2+2^{-2j})^{\frac{m}{2}}}{\sinh^m(2^{-j-1})}\left|\frac{\sin(\frac{\xi}{2})}{\frac{\xi}{2}}\right|^N \\ &\leq \frac{C_1'}{(1+2^{2j}\xi^2)^{\frac{m}{2}}(1+|\xi|)^N} \end{aligned}$$

and, for $j < 0$,

$$\begin{aligned} |\widehat{\varphi}^{(j),(N)}(\xi)| &\leq \frac{1}{(1+2^{2j}\xi^2)^m}\left(\frac{\cosh(2^{-j})+1}{2\sinh^2(2^{-j-1})}\right)^{\frac{m}{2}}\left|\frac{\sin(\frac{\xi}{2})}{\frac{\xi}{2}}\right|^N \\ &\leq \frac{C_2}{(1+2^{2j}\xi^2)^{\frac{m}{2}}(1+|\xi|)^N} \end{aligned}$$

Let us consider now the second estimation. Let $\varepsilon > 0$. Similarly to ([10]), there is $c_\varepsilon > 0$ such that

$$\prod_{p\in\mathbb{N}_0} P^{(j+p)}_{\frac{N+\tilde N}{2},m}\left(\sin^2(2^{-p-1}\xi)\right) \leq c_\varepsilon 2^{-m\inf\{0,j\}}(1+|\xi|)^{\varepsilon+(\tilde N+N+2m-2)\frac{\ln(3)}{2\ln(2)}}.$$

So, for $j \geq 0$, we have

$$\begin{aligned} |\widehat{\widetilde{\varphi}}^{(j),(N,\tilde N)}(\xi)| &\leq c_\varepsilon'\left(\frac{2^{j+1}}{1+2^{2j}\xi^2}\right)^m \frac{(\xi^2+2^{-2j})^{\frac{m}{2}}}{\cosh^m(2^{-j-1})}(1+|\xi|)^{-\tilde N+\varepsilon+(\tilde N+N+2m-2)\frac{\ln(3)}{2\ln(2)}} \\ &\leq \frac{c_\varepsilon''}{(1+2^{2j}\xi^2)^{\frac{m}{2}}(1+|\xi|)^{\tilde N-(\tilde N+N+2m-2)\frac{\ln(3)}{2\ln(2)}-\varepsilon}} \end{aligned}$$

and, for $j < 0$,

$$
\begin{aligned}
&|\widehat{\widetilde{\varphi}}^{(j),(N,\tilde N)}(\xi)| \\
&\leq c'_\varepsilon 2^{-mj}\left(\frac{2^{j+1}}{1+2^{2j}\xi^2}\right)^m\left(\frac{\cosh(2^{-j})-\cos(\xi)}{2\cosh^2(2^{-j-1})}\right)^{\frac{m}{2}}\left|\frac{\sin(\frac{\xi}{2})}{\frac{\xi}{2}}\right|^{\tilde N}(1+|\xi|)^{\varepsilon+(\tilde N+N+2m-2)\frac{\ln(3)}{2\ln(2)}} \\
&\leq \frac{c''_\varepsilon}{(1+2^{2j}\xi^2)^{\frac{m}{2}}(1+|\xi|)^{\tilde N-(\tilde N+N+2m-2)\frac{\ln(3)}{2\ln(2)}-\varepsilon}}
\end{aligned}
$$

c) As in ([10]), the infinite product

$$\prod_{p\in\mathbb{N}_0} P^{(j+p)}_{\frac{N+\tilde N}{2},m}\left(\sin^2(2^{-p-1}\xi)\right)$$

defines an entire function with exponential growth independent of j. It follows then from their definitions and from the Paley-Wiener theorem that $\psi^{(j)}$ and $\widetilde{\psi}^{(j)}$ have compact support independent of j.

The symmetry axis comes from the fact that

$$
\begin{aligned}
\widehat{\psi}^{(j),(N,\tilde N)}(-\xi) &= (-1)^{\tilde N}e^{i\xi}\widehat{\psi}^{(j),(N,\tilde N)}(\xi) \\
\widehat{\widetilde{\psi}}^{(j),(N,\tilde N)}(-\xi) &= (-1)^{N}e^{i\xi}\widehat{\widetilde{\psi}}^{(j),(N,\tilde N)}(\xi).
\end{aligned}
$$

From proposition 1.22, we already know that the families $\psi_{j,k}^{(N,\tilde N)}$ $(j,k\in\mathbb{Z})$ and $\widetilde{\psi}_{j,k}^{(N,\tilde N)}$ $(j,k\in\mathbb{Z})$ are frames for $H^m(\mathbb{R})$. To get that they are Riesz bases, it suffices then to prove that $\langle\psi_{j,k}^{(N,\tilde N)}, \widetilde{\psi}_{j',k'}^{(N,\tilde N)}\rangle = \delta_{j,j'}\delta_{k,k'}$. This last property is clearly satisfied. Indeed, fix $j\in\mathbb{Z}$ and for every $J\in\mathbb{N}_0$, define the function $F_J^{(j)}$ as follows

$$F_J^{(j)}(\xi) := \chi_{[-\pi,\pi]}(2^{-J}\xi)\prod_{p=1}^{J} m_0^{(j+p),(N)}(2^{-p}\xi)\overline{\widetilde{m_0}^{(j+p),(N,\tilde N)}(2^{-p}\xi)}.$$

Using (1.3), we get

$$\int_{\mathbb{R}} e^{ik\xi}F_J^{(j)}(\xi)d\xi = 2\pi\delta_{k,0}. \tag{1.5}$$

As in ([9]), we can take the limit inside the integral because of the special choice of the filters; hence we get $\langle\varphi_{j,k}^{(N)}, \widetilde{\varphi}_{j,k'}^{(N,\tilde N)}\rangle_{H^m(\mathbb{R})} = \delta_{k,k'}$. The result on orthogonality between the wavelets in obtained as in [15] (Lemma 3.7 on page 507). ∎

In $H^1(\mathbb{R})$, the figure "**Fig 1**" gives some pictures of $\psi^{(-1),(1,1)}$ and $\widetilde{\psi}^{(-1),(1,1)}$.

Remark 1.26 *Let us consider the previous c) without the assumption on $N,\tilde N$. Using Fatou's theorem in (1.5), we obtain that the product*

$$(1+2^{2j}\xi^2)^m\widehat{\varphi}^{(j),(N)}\,\overline{\widehat{\widetilde{\varphi}}^{(j),(N,\tilde N)}}$$

belongs to $L^1(\mathbb{R})$. Hence, taking the limit inside the integral, we get also orthogonality; but of course it makes sense only if one considers the product of the two functions (it is not necessarily the scalar product of two elements belonging to $H^m(\mathbb{R})$). Hence we still obtain

$$\langle \psi_{j,k}^{(N,\tilde{N})}, \widetilde{\psi}_{j',k'}^{(N,\tilde{N})} \rangle_{H^m(\mathbb{R})} = \delta_{j,j'}\delta_{k,k'}.$$

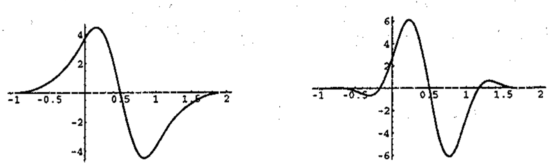


Fig 1

1.2.5 Discussion on the filters

The aim is to use the multiresolution analysis point of view and the infinite product construction to obtain compactly supported regular biorthogonal wavelets. We are then led (see [15],[9],[11]) to take polynomial filters containing the "good" factors $\frac{1+e^{-i\xi}}{2}$, $\cos(\frac{\xi+i2^{-j}}{2})$. If we decide to take the first filter $m_0^{(j),(N)}(\xi)$, we look for a second one $\widetilde{m_0}^{(j),(N,\tilde{N})}(\xi)$ containing also the "good" factors and another factor so that (1.3) is satisfied. Our particular choice for symmetry comes from the discussion in ([15]) and from the results of ([9]). Other filters could be found; but we don't have general results on them to achieve the construction of biorthogonal wavelets.

1.3 Construction of biorthogonal bases of wavelets in $H^m(\mathbb{R})$ with orthogonality between levels

We want to show that starting from the same scaling functions $\varphi^{(j)}$, it is possible to construct biorthogonal wavelets $\psi_{j,k}(x) = 2^{j/2}\psi^{(j)}(2^jx - k)$ $(j,k \in \mathbb{Z})$ and $\widetilde{\psi}_{j,k} = 2^{j/2}\widetilde{\psi}^{(j)}(2^jx-k)$ $(j,k \in \mathbb{Z})$ in $H^m(\mathbb{R})$ such that $\psi^{(j)}(x/2)$ is an exponential spline (i.e. a regular function on $\mathbb{R}$ which, on an interval of type $[k, k+1]$, $k \in \mathbb{Z}$, is equal to a function of type $e^{ax}P(x)$, P polynomial, $a \in \mathbb{R}$) with compact support satisfying $\langle \psi_{j,k}, \psi_{j',k'} \rangle = 0$ if $j \neq j'$, and $\widetilde{\psi}^{(j)}$ an exponential spline satisfying the same condition. This is a Sobolev version of the construction of Chui and Wang in ([13]).

We recall that take the filters

$$m_0^{(j),(N)}(\xi) = (M^{(j)}(\xi))^m \cos^N(\frac{\xi}{2})e^{-i\mu\xi/2}$$

will yield the fathers

$$\widehat{\varphi}^{(j),(N)}(\xi) = \frac{1}{(1+i2^j\xi)^m}\prod_{p=1}^{+\infty} m_0^{(j+p),(N)}(2^{-p}\xi) = \frac{(-i)^m e^{i(m-\mu)\xi/2}}{(1+2^{2j}\xi^2)^m}\frac{\sinh^m(\frac{\xi+i2^{-j}}{2})}{\sinh^m(2^{-j-1})}\left(\frac{\sin(\frac{\xi}{2})}{\frac{\xi}{2}}\right)^N.$$

We define the function $\omega^{(j),(N)}$ as follows

$$\omega^{(j),(N)}(\xi) := \sum_{k\in\mathbb{Z}}(1+2^{2j}(\xi+2k\pi)^2)^m|\widehat{\varphi}^{(j),(N)}(\xi+2k\pi)|^2.$$

Proposition 1.27 *For every $j \in \mathbb{Z}$, the function $\omega^{(j),(N)}$ is a 2π-periodic trigonometric polynomial; its degree is independent of j. Moreover, there are constants $A, B > 0$ such that*

$$A \le \omega^{(j),(N)}(\xi) \le B, \quad \forall\xi\in\mathbb{R},\ \forall j\in\mathbb{Z}.$$

PROOF. See [6]. ∎

Corollary 1.28 *For every integer j, the functions $\varphi_{j,k}^{(N)}$ $(k \in \mathbb{Z})$ satisfy the Riesz condition in $H^m(\mathbb{R})$.* ∎

We denote by $V_j^{(N)}$ the closed linear hull of the functions $\varphi_{j,k}^{(N)}$, $(k\in\mathbb{Z})$. We have $V_j^{(N)} \subset V_{j+1}^{(N)}$ for every $j\in\mathbb{Z}$.

Proposition 1.29 *Let $j \in \mathbb{Z}$. In $H^m(\mathbb{R})$, the dual basis of the Riesz basis $\{\varphi_{j,k}^{(N)} : k\in\mathbb{Z}\}$ is given by $\{\widetilde{\varphi}_{j,k}^{(N)} = 2^{\frac{j}{2}}\widetilde{\varphi}^{(j),(N)}(2^j\cdot - k) : k\in\mathbb{Z}\} \subset V_j^{(N)}$ where the function $\widetilde{\varphi}^{(j),(N)}$ is defined by*

$$\widehat{\widetilde{\varphi}}^{(j),(N)} := \frac{\widehat{\varphi}^{(j),(N)}}{\omega^{(j),(N)}}.$$

Taking all j, those functions $\widetilde{\varphi}_{j,k}^{(N)}$, $j\in\mathbb{Z}$, satisfy the scaling relations

$$\widehat{\widetilde{\varphi}}^{(j),(N)}(2\xi) = \widetilde{m_0}^{(j+1),(N)}(\xi)\,\widehat{\widetilde{\varphi}}^{(j+1),(N)}(\xi)$$

with

$$\widetilde{m_0}^{(j),(N)}(\xi) := \frac{\omega^{(j),(N)}(\xi)\, m_0^{(j),(N)}(\xi)}{\omega^{(j-1),(N)}(2\xi)}.$$

∎

For $j\in\mathbb{Z}$, we define $W_j^{(N)}$ as the orthogonal complement in $H^m(\mathbb{R})$ of $V_j^{(N)}$ in $V_{j+1}^{(N)}$. We seek for a basis of $W_j^{(N)}$ of type $\psi_{j,k}^{(N)}(x) = 2^{j/2}\psi^{(j),(N)}(2^jx-k)$, $k\in\mathbb{Z}$, where $\psi^{(j),(N)}$ has a compact support independent of j. We are then led to the following results.

Proposition 1.30 (Description of $W_j^{(N)}$) *a) We have*

$$W_j^{(N)} = \{f \in H^m(\mathbb{R}) : \ \exists\, \lambda \in L^2_{\text{loc}}(\mathbb{R}),\ \pi\text{-per.} : \\ \widehat{f}(2^{j+1}\xi) = e^{i\xi}\lambda(\xi)\ \omega^{(j+1),(N)}(\xi+\pi)\,\overline{m_0^{(j+1),(N)}}(\xi+\pi)\widehat{\varphi}^{(j+1),(N)}(\xi)\ \text{a.e.}\}.$$

b) If

$$E^{(j),(N)}(\xi) := \sum_{k\in\mathbb{Z}} (1+2^{2j}(\xi+2k\pi)^2)^m |\widehat{\psi}^{(j),(N)}(\xi+2k\pi)|^2$$

where

$$\widehat{\psi}^{(j),(N)}(2\xi) := -e^{-i\xi}\,\overline{m_0^{(j+1),(N)}}(\xi+\pi)\ \omega^{(j+1),(N)}(\xi+\pi)\ \widehat{\varphi}^{(j+1),(N)}(\xi)$$

then we have

$$E^{(j),(N)}(2\xi) = \omega^{(j+1),(N)}(\xi)\,\omega^{(j+1),(N)}(\xi+\pi)\,\omega^{(j),(N)}(2\xi).$$

PROOF. See [6]. ∎

Proposition 1.31 *a) For every $j \in \mathbb{Z}$, the functions $\psi_{j,k}^{(N)} = 2^{j/2}\psi^{(j),(N)}(2^j\cdot - k)$, $k \in \mathbb{Z}$, form a Riesz basis for $W_j^{(N)} = V_{j+1}^{(N)} \cap V_j^{(N)\perp}$ and the bounds are independent of the level j.*

b) For every j, the function $\psi^{(j),(N)}(\frac{\cdot}{2})$ is an exponential spline with compact support.

PROOF. See [6]. ∎

From this result we obtain directly the following one.

Theorem 1.32 *The functions $\psi_{j,k}^{(N)}$ $(j,k\in\mathbb{Z})$ are a Riesz basis for $H^m(\mathbb{R})$.*

PROOF. See [6]. ∎

Moreover, the Riesz basis $\psi_{j,k}^{(N)}$ $(j,k\in\mathbb{Z})$ and its dual basis have an interpretation in terms of signal analysis, as it is already the case for the previous construction.

Proposition 1.33 *The dual basis is $\widetilde{\psi}_{j,k}^{(N)}(x) = 2^{j/2}\widetilde{\psi}^{(j),(N)}(2^jx-k)$ $(j,k\in\mathbb{Z})$, where*

$$\widehat{\widetilde{\psi}}^{(j),(N)}(\xi) := \frac{\widehat{\psi}^{(j),(N)}(\xi)}{E^{(j),(N)}(\xi)}$$

We also have

$$\begin{aligned} \widehat{\widetilde{\psi}}^{(j),(N)}(2\xi) &= \widetilde{m_1}^{(j+1),(N)}(\xi)\,\widehat{\widetilde{\varphi}}^{(j+1),(N)}(\xi) \\ \widehat{\psi}^{(j),(N)}(2\xi) &= m_1^{(j+1),(N)}(\xi)\,\widehat{\varphi}^{(j+1),(N)}(\xi) \end{aligned}$$

with

$$\widetilde{m_1}^{(j),(N)}(\xi) \;:=\; -e^{-i\xi}\,\frac{\overline{m_0^{(j),(N)}}(\xi+\pi)}{\omega^{(j-1),(N)}(2\xi)}$$

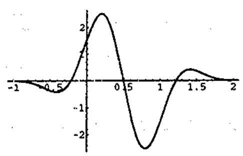
$$m_1^{(j),(N)}(\xi) \;:=\; -e^{-i\xi}\,\overline{m_0^{(j),(N)}}(\xi+\pi)\;\omega^{(j),(N)}(\xi+\pi).$$

The filters $m_0^{(j),(N)}$, $\widetilde{m_0}^{(j),(N)}$, $m_1^{(j),(N)}$, $\widetilde{m_1}^{(j),(N)}$ *satisfy*

$$\begin{cases} \widetilde{m_0}^{(j),(N)}(\xi)\,\overline{m_0^{(j),(N)}}(\xi) + \widetilde{m_1}^{(j),(N)}(\xi)\,\overline{m_1^{(j),(N)}}(\xi) & = 1 \\ \widetilde{m_0}^{(j),(N)}(\xi)\,\overline{m_0^{(j),(N)}}(\xi+\pi) + \widetilde{m_1}^{(j),(N)}(\xi)\,\overline{m_1^{(j),(N)}}(\xi+\pi) & = 0. \end{cases}$$

PROOF. See [6]. ∎

In $H^1(\mathbb{R})$, the figure "**Fig. 2**" gives some pictures of $\psi^{(-1),(1)}$ and $\widetilde{\psi}^{(-1),(1)}$.



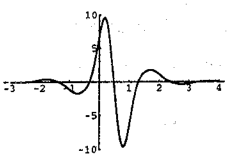


Fig. 2

Chapter 2

Construction of wavelets bases in $H^s_{1-\mathrm{per}}(\mathbb{R})$

The aim of the second chapter is to present some constructions of wavelets Riesz bases in the 1-periodic Sobolev spaces $H^s_{1-\mathrm{per}}(\mathbb{R})$ $(s \in \mathbb{R})$. We start by an introduction to these spaces. A relation between periodic Sobolev spaces and local Sobolev spaces generalizing what happens in $L^2_{1-\mathrm{per}}(\mathbb{R})$ is there given.

2.1 Periodic Sobolev spaces

Definition 2.1 *Let $s \in \mathbb{R}$. We define the Sobolev space $H^s_{1-\mathrm{per}}(\mathbb{R})$ as the linear space of all 1-periodic distributions u on $\mathbb{R}$ such that*

$$\sum_{k\in\mathbb{Z}_0} |k|^{2s}|u_k|^2 < +\infty$$

where u_k $(k \in \mathbb{Z})$ is the k-th Fourier coefficient of u. Remind that if u is a 1-periodic distribution, then the sequence $(u_k)_{k\in\mathbb{Z}}$ of its Fourier coefficients is defined as the unique sequence of $\mathbb{C}$ such that

$$u(\varphi) = \sum_{k\in\mathbb{Z}} u_k \int_{\mathbb{R}} \varphi(x)e^{-2i\pi kx}dx, \quad \forall\varphi \in D(\mathbb{R}).$$

Endowed with the natural scalar product

$$\langle u\,,\, v\rangle_{H^s_{1-\mathrm{per}}(\mathbb{R})} = u_0\,\overline{v_0} + (u,v)_{H^s_{1-\mathrm{per}}(\mathbb{R})} = u_0\,\overline{v_0} + \sum_{k\in\mathbb{Z}_0} |k|^{2s}u_k\,\overline{v_k}\,,$$

$H^s_{1-\mathrm{per}}(\mathbb{R})$ is a Hilbert space. We denote by $\|\cdot\|_{H^s_{1-\mathrm{per}}(\mathbb{R})}$ the corresponding norm

$$\|u\|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})} = |u_0|^2 + |u|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})} = |u_0|^2 + \sum_{k\in\mathbb{Z}_0} |k|^{2s}|u_k|^2.$$

Remark 2.2 *The norm* $\|\cdot\|_s^{(1)}$ *(resp.* $\|\cdot\|_s^{(2)}$*) given by*

$$\|u\|_s^{(1)} = \sqrt{\sum_{k\in\mathbb{Z}}(1+k^2)^s|u_k|^2}$$

$$\left(\text{resp.}\quad \|u\|_s^{(2)} = \sqrt{\sum_{k\in\mathbb{Z}}(1+|k|^{2s})|u_k|^2}\ \right)$$

is equivalent to $\|\cdot\|_{H^s_{1-\mathrm{per}}(\mathbb{R})}$ *on* $H^s_{1-\mathrm{per}}(\mathbb{R})$ *for* $s\in\mathbb{R}$ *(resp.* $s\in\,]0,+\infty[$*) and, if* $s\in\,]\frac{1}{2},+\infty[$, *we can also mention*

$$\|u\|_s = \sqrt{|u(0)|^2 + \sum_{k\in\mathbb{Z}_0}|k|^{2s}|u_k|^2}$$

which is related to the scalar product

$$\langle u\,,\,v\rangle_s = u(0)\,\overline{v(0)} + \sum_{k\in\mathbb{Z}_0}|k|^{2s}u_k\,\overline{v_k}\,.$$

It is also well known (see [39]) that for every $s\geq 0$ *such that* $s\notin\frac{1}{2}+\mathbb{N}$, *this norm* $\|\cdot\|_{H^s_{1-\mathrm{per}}(\mathbb{R})}$ *is equivalent to the norm* $|\!|\!|\cdot|\!|\!|_{H^s_{1-\mathrm{per}}(\mathbb{R})}$ *defined by*

$$|\!|\!|u|\!|\!|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})} = \begin{cases} \|u\|^2_{L^2_{1-\mathrm{per}}(\mathbb{R})} + \|\mathrm{D}^m u\|^2_{L^2_{1-\mathrm{per}}(\mathbb{R})} & \text{if } s=m\in\mathbb{N} \\ \|u\|^2_{L^2_{1-\mathrm{per}}(\mathbb{R})} + \int_0^1 dy\int_0^1 dx\frac{|\mathrm{D}^m u(x)-\mathrm{D}^m u(y)|^2}{|x-y|^{1+2\sigma}} & \text{if } s=m+\sigma, \\ & m\in\mathbb{N},\ \sigma\in\,]0,1[\,. \end{cases}$$

In what concerns the duality between $H^s_{1-\mathrm{per}}(\mathbb{R})$ and $H^{-s}_{1-\mathrm{per}}(\mathbb{R})$, we have the following result.

Proposition 2.3 *Let* $s,\alpha\in\mathbb{R}$. *The functional*

$$u\longmapsto\langle u\,,\,v\rangle_{H^s_{1-\mathrm{per}}(\mathbb{R})}$$

is bounded on $H^{s+\alpha}_{1-\mathrm{per}}(\mathbb{R})$ *if and only if* $v\in H^{s-\alpha}_{1-\mathrm{per}}(\mathbb{R})$. ∎

The next proposition gives a characterization of the spaces $H^s_{1-\mathrm{per}}(\mathbb{R})$.

Proposition 2.4 *For every real number* s, *we have*

$$H^s_{1-\mathrm{per}}(\mathbb{R}) = \{u\in H^s_{\mathrm{loc}}(\mathbb{R}):\ u \text{ is } 1\text{-per.}\}$$

PROOF OF PROPOSITION 2.4 FOR $s \geq 0$

The result is clear for $s = m \in \mathbb{N}$.

Let us consider the case $s \in]0, 1[$.
a) Let u be a 1-periodic function of $H^s_{\mathrm{loc}}(\mathbb{R})$. We have to prove that u belongs to $H^s_{1-\mathrm{per}}(\mathbb{R})$. Let K be the compact interval $[-2, 2]$ of $\mathbb{R}$ and φ a function of $D(\mathbb{R})$ which is equal to 1 on K. Using φ in Proposition 1.5, one gets

$$\frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} \in L^1(K \times K).$$

From Fubini theorem, we can deduce that for almost every $x \in [-1, 1]$, the function

$$\frac{|u(x + y) - u(y)|^2}{|x|^{1+2s}}$$

belongs to $L^1(]0, 1[)$ and that

$$F(x) := \int_0^1 \frac{|u(x + y) - u(y)|^2}{|x|^{1+2s}} dy = \frac{1}{|x|^{1+2s}} \sum_{k\in\mathbb{Z}} |u_k|^2 \, |1 - e^{2i\pi kx}|^2$$

belongs to $L^1(]-1, 1[)$. Because of integrability far from 0, we finally get that the function F belongs to $L^1(\mathbb{R})$. So, the function u belongs to $H^s_{1-\mathrm{per}}(\mathbb{R})$ since we have

$$\int_{\mathbb{R}} F(x)dx = 8 \sum_{k\in\mathbb{Z}} |u_k|^2 \int_0^{+\infty} \frac{\sin^2(\pi|k|x)}{x^{1+2s}} dx = C \sum_{k\in\mathbb{Z}_0} |k|^{2s} |u_k|^2.$$

b) Let $u \in H^s_{1-\mathrm{per}}(\mathbb{R})$ and $\varphi \in D(\mathbb{R})$. We have to prove that the function

$$F(x, y) := \frac{|(u\varphi)(x) - (u\varphi)(y)|^2}{|x - y|^{1+2s}}$$

belongs to $L^1(\mathbb{R}^2)$. Let $N \in \mathbb{N}_0$ such that $\mathrm{supp}(\varphi) \subset [-N, N]$ and $I := [-N - 1, N + 1]$. On one hand, the function F belongs to $L^1(\mathbb{R} \times (\mathbb{R} \setminus I))$ because if $x \in \mathrm{supp}(\varphi)$ and $y \in \mathbb{R} \setminus I$, then we can write

$$F(x, y) \leq \frac{|(u\varphi)(x)|^2}{(|y| - N)^{1+2s}}.$$

On the other hand, let us prove that the function F also belongs to $L^1(\mathbb{R} \times I)$. On $\mathbb{R} \times I$, we can write

$$F(x, y) \leq 2F_1(x, y) + 2F_2(x, y)$$

with

$$F_1(x, y) := \frac{|u(x) - u(y)|^2 \, |\varphi(x)|^2}{|x - y|^{1+2s}}, \quad F_2(x, y) := \frac{|u(y)|^2 \, |\varphi(x) - \varphi(y)|^2}{|x - y|^{1+2s}}.$$

The function F_1 belongs to $L^1(\mathbb{R} \times I)$ since the function

$$\frac{|u(x+y)-u(y)|^2}{|x|^{1+2s}}$$

belongs to $L^1(\mathbb{R} \times I)$. Indeed, for x fixed in $\mathbb{R}$, we have

$$\int_I |u(x+y)-u(y)|^2 dy = 2(N+1)\int_0^1 |u(x+y)-u(y)|^2 dy = 2(N+1)\sum_{k\in\mathbb{Z}} |u_k|^2\,|1-e^{2i\pi kx}|^2$$

and now the function

$$\frac{1}{|x|^{1+2s}}\int_I |u(x+y)-u(y)|^2 dy$$

belongs to $L^1(\mathbb{R})$ since, $\forall k \in \mathbb{Z}_0$, we have

$$\int_{\mathbb{R}} \frac{|1-e^{2i\pi kx}|^2}{|x|^{1+2s}} dx = 8|k|^{2s}\int_0^{+\infty} \frac{\sin^2(\pi x)}{x^{1+2s}}\, dx.$$

In what concerns the function F_2, we have

$$\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}} \le \begin{cases} \frac{C'}{|x-y|^{1+2s}} & \text{if } \; |x-y|\ge 1 \\ \frac{C'}{|x-y|^{2s-1}} & \text{if } \; |x-y| \le 1 \end{cases}$$

so we get

$$\int_{\mathbb{R}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{1+2s}} dx \le C''$$

and we conclude since $u \in L^2_{\mathrm{loc}}(\mathbb{R})$.

The case $s \in \,]0,+\infty[$ can be easily deduced from the case $s \in \,]0,1[$. Indeed, if $s \in \,]0,+\infty[\, \setminus \mathbb{N}_0$, we can write $s = m+\sigma$ with $m \in \mathbb{N}$, $\sigma \in \,]0,1[$ and we have

$$\begin{aligned} &\{u \in H^s_{\mathrm{loc}}(\mathbb{R}) : \; u \text{ is } 1\text{-per}\} \\ &= \{u \in H^m_{\mathrm{loc}}(\mathbb{R}) : \; u \text{ is } 1\text{-per and } \mathrm{D}^m u \in H^\sigma_{\mathrm{loc}}(\mathbb{R})\} \\ &= \{u \in H^m_{1-\mathrm{per}}(\mathbb{R}) : \; \mathrm{D}^m u \in H^\sigma_{1-\mathrm{per}}(\mathbb{R})\} \\ &= H^s_{1-\mathrm{per}}(\mathbb{R}). \end{aligned}$$

∎

Before presenting the case $s < 0$, let us present a norm equivalence between the Sobolev spaces on $\mathbb{R}$ and the periodic ones.

Proposition 2.5 *Let $s \in [0,+\infty[$ and F be a positive function of $D(\mathbb{R})$. There is a constant $C > 0$ such that*

$$\|Fu\|_{H^s(\mathbb{R})} \le C\|u\|_{H^s_{1-\mathrm{per}}(\mathbb{R})}, \quad \forall u \in H^s_{1-\mathrm{per}}(\mathbb{R}).$$

and, if $s \in \,]0,1[$ and if F equals 1 in a neighborhood of $[0,1]$, then there is also a constant $c > 0$ such that

$$c\|u\|_{H^s_{1-\mathrm{per}}(\mathbb{R})} \le \|Fu\|_{H^s(\mathbb{R})}, \quad \forall u \in H^s_{1-\mathrm{per}}(\mathbb{R}).$$

PROOF. Let $s \in [0,+\infty[$ and F be a positive function of $D(\mathbb{R})$.

On one hand, we get, $\forall u \in H^s_{1-\mathrm{per}}(\mathbb{R})$,

$$\|Fu\|^2_{H^s(\mathbb{R})} \leq \begin{cases} C\sum_{\alpha=0}^m \|\mathrm{D}^\alpha(Fu)\|^2_{L^2(\mathbb{R})} & \text{if } s=m\in\mathbb{N} \\ C\Big(\sum_{\alpha=0}^m \|\mathrm{D}^\alpha(Fu)\|^2_{L^2(\mathbb{R})} + \int_\mathbb{R} dx \int_\mathbb{R} dy \, \frac{|\mathrm{D}^m(Fu)(x)-\mathrm{D}^m(Fu)(y)|^2}{|x-y|^{1+2\sigma}}\Big) \\ \quad \text{if } s=m+\sigma,\ (m\in\mathbb{N},\sigma\in]0,1[) \end{cases}$$

For every $\alpha \in \{0,\dots,m\}$, we can write

$$\begin{aligned} \|\mathrm{D}^\alpha(Fu)\|^2_{L^2(\mathbb{R})} &\leq \sum_{j=0}^\alpha C^j_\alpha \sum_{(\ell\in\mathbb{Z})} \int_0^1 dy |(\mathrm{D}^{\alpha-j}F)(y+\ell)|^2 |(\mathrm{D}^j u)(y)|^2 \\ &\leq \sum_{j=0}^\alpha C^j_\alpha C_{\alpha-j} \|\mathrm{D}^j u\|^2_{L^2(]0,1[)} \leq C\, \|u\|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})} \end{aligned}$$

and, using the same estimations as in the point b) of the proof of Proposition 2.4, we get also

$$\int_\mathbb{R} dx \int_\mathbb{R} dy \, \frac{|\mathrm{D}^m(Fu)(x)-\mathrm{D}^m(Fu)(y)|^2}{|x-y|^{1+2\sigma}} \leq C\|u\|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})}.$$

On the other hand, we suppose that $s \in]0,1[$ and that F equals 1 in a neighborhood of $[0,1]$. We get

$$\begin{aligned} \|u\|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})} &= |u_0|^2 + \sum_{k\in\mathbb{Z}_0} |k|^{2s}|u_k|^2 \\ &= |u_0|^2 + C\sum_{k\in\mathbb{Z}_0} \int_0^{+\infty} dx \, \frac{\sin^2(kx)}{x^{1+2s}} |u_k|^2 \\ &= |u_0|^2 + \frac{C}{4}\sum_{k\in\mathbb{Z}_0} \int_0^{+\infty} dx \, \frac{1}{x^{1+2s}} |u_k - (u(\cdot+x))_k|^2 \\ &= |u_0|^2 + \frac{C}{4}\int_0^{+\infty} dx \, \frac{1}{x^{1+2s}} \|u-u(\cdot+x)\|^2_{L^2(]0,1[)} \\ &\leq |u_0|^2 + \frac{C}{2}\int_0^{+\infty} dx \, \frac{1}{x^{1+2s}} \int_0^1 dy \, |(Fu)(y)-(Fu)(x+y)|^2 \\ &\quad + \frac{C}{2}\int_0^{+\infty} dx \, \frac{1}{x^{1+2s}} \int_0^1 dy \, |F(x+y)-1|^2 \, |u(x+y)|^2. \end{aligned}$$

Therefore we obtain the inequality because we have

$$\begin{aligned} &|u_0|^2 + \frac{C}{2}\int_0^{+\infty} dx \, \frac{1}{x^{1+2s}} \int_0^1 dy \, |(Fu)(y)-(Fu)(x+y)|^2 \\ &\leq |u_0|^2 + \frac{C}{2}\int_\mathbb{R} dx \, \frac{1}{|x|^{1+2s}} \int_\mathbb{R} dy \, |(Fu)(y)-(Fu)(x+y)|^2 \\ &\leq \|Fu\|^2_{L^2(\mathbb{R})} + \frac{C}{2}\int_\mathbb{R} dz \int_\mathbb{R} dy \, \frac{|(Fu)(y)-(Fu)(z)|^2}{|z-y|^{1+2s}} \\ &\leq C'\|Fu\|^2_{H^s(\mathbb{R})} \end{aligned}$$

and

$$\int_0^{+\infty} dx\, \frac{1}{x^{1+2s}} \int_0^1 dy\, |F(x+y)-1|^2\, |u(x+y)|^2$$
$$= \int_\varepsilon^{+\infty} dx\, \frac{1}{x^{1+2s}} \int_0^1 dy\, |F(x+y)-1|^2\, |u(x+y)|^2$$
$$\leq C' \|u\|^2_{L^2(]0,1[)} \leq C' \|Fu\|^2_{L^2(\mathbb{R})}$$

for an $\varepsilon > 0$. ∎

PROOF OF PROPOSITION 2.4 FOR $s < 0$

Let $u \in H^s_{1-\mathrm{per}}(\mathbb{R})$, $\varphi \in D(\mathbb{R})$ and F be a positive function of $D(\mathbb{R})$ equals to 1 in a neighborhood of $[0,1]$. Using Proposition 2.5 and the map S defined on $D(\mathbb{R})$ by

$$S:\ \psi \mapsto \sum_{\ell\in\mathbb{Z}} \psi(\cdot + \ell),$$

we get

$$\begin{aligned}
|(\varphi u)(\phi)|^2 &= \left| \sum_{k\in\mathbb{Z}} u_k (S(\varphi\phi))_k \right|^2 \\
&\leq \|u\|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})}\, \|S(\varphi\phi)\|^2_{H^{m+\sigma}_{1-\mathrm{per}}(\mathbb{R})} \\
&\leq C\|u\|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})} \left(|(S(\varphi\phi))_0|^2 + \|\mathrm{D}^m(S(\varphi\phi))\|^2_{H^\sigma_{1-\mathrm{per}}(\mathbb{R})} \right) \\
&\leq C'\|u\|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})} \left(\|\varphi\|^2_{L^2(\mathbb{R})}\, \|\phi\|^2_{L^2(\mathbb{R})} + \|F\mathrm{D}^m\left(S(\varphi\phi)\right)\|^2_{H^\sigma(\mathbb{R})} \right),
\end{aligned}$$

$\forall \phi \in D(\mathbb{R})$, if $m \in \mathbb{N}$ and $\sigma \in]0,1[$ are such that $-s = m + \sigma$. Then we can easily obtain that there is a constant C_1 such that

$$|(\varphi u)(\phi)| \leq C_1 \|\phi\|_{H^{-s}(\mathbb{R})}, \quad \forall \phi \in D(\mathbb{R}).$$

Indeed, we have

$$\begin{aligned}
\|F\mathrm{D}^m\left(S(\varphi\phi)\right)\|_{H^\sigma(\mathbb{R})} &= \|F(\cdot) \sum_{(\ell\in\mathbb{Z})} \sum_{j=0}^{m} C^j_m \mathrm{D}^{m-j}\varphi(\cdot+\ell)\mathrm{D}^j\phi(\cdot+\ell)\|_{H^\sigma(\mathbb{R})} \\
&\leq \sum_{(\ell\in\mathbb{Z})} \sum_{j=0}^{m} C^j_m C_{m-j} \|\mathrm{D}^j\phi(\cdot+\ell)\|_{H^\sigma(\mathbb{R})} \\
&\leq C\|\phi\|_{H^{-s}(\mathbb{R})}, \quad \forall \phi \in D(\mathbb{R}).
\end{aligned}$$

Now, in order to obtain the second part of the proof, let u be a 1-periodic distribution of $H^s_{\mathrm{loc}}(\mathbb{R})$. Let also G be a positive function of $D(\mathbb{R})$ satisfying $\sum_{\ell\in\mathbb{Z}} G(\cdot+\ell) \equiv 1$ and $F_0 \in D(\mathbb{R})$ such that $F_0 \equiv 1$ in a neighborhood of $\mathrm{supp}(G)$. To prove that $u \in H^s_{1-\mathrm{per}}(\mathbb{R})$, one has merely to prove that

$$\left| \sum_{k\in\mathbb{Z}_0} u_k v_k \right|^2 \leq C\Big(\sum_{k\in\mathbb{Z}_0} |k|^{-2s} |v_k|^2 \Big)$$

for every $v \in H^{-s}_{1-\mathrm{per}}(\mathbb{R})$. Since $F_0 u \in H^s(\mathbb{R})$ and using again Proposition 2.5, we get

$$\begin{aligned}
\Big|\sum_{k\in\mathbb{Z}_0} u_k v_k\Big| &= \Big|u_{(x)}\Big(G(x)\sum_{k\in\mathbb{Z}_0} e^{2i\pi kx} v_k\Big)\Big| \\
&= \Big|(F_0 u)_{(x)}\Big(G(x)\sum_{k\in\mathbb{Z}_0} e^{2i\pi kx} v_k\Big)\Big| \\
&\leq C\Big\|G\sum_{k\in\mathbb{Z}_0} e^{2i\pi k\cdot} v_k\Big\|_{H^{-s}(\mathbb{R})} \\
&\leq C'\Big\|\sum_{k\in\mathbb{Z}_0} e^{2i\pi k\cdot} v_k\Big\|_{H^{-s}_{1-\mathrm{per}}(\mathbb{R})} \\
&= C'\sqrt{\sum_{k\in\mathbb{Z}_0} |k|^{-2s}|v_k|^2}, \quad \forall v \in H^{-s}_{1-\mathrm{per}}(\mathbb{R}).
\end{aligned}$$

The proposition is proved. ∎

2.2 Spline spaces

Let $[a,b]$ be a compact interval of $\mathbb{R}$, $N \in \mathbb{N}_0$, and

$$\Delta \equiv \quad a = x_0 < x_1 < \ldots < x_N = b$$

a mesh on the interval $[a,b]$.

Notation 2.6 *Let $r \in \mathbb{N}_0$. We denote by $S^r([a,b],\Delta)$ the set of smoothest splines of degree r on $[a,b]$, with respect to the mesh Δ. Remind that for $r = 0$, it represents the set of functions defined in $[a,b]$ which are constant on $[x_j, x_{j+1}[$, for every $j \in \{0,\ldots,N-1\}$. For $r \in \mathbb{N}_0$, it represents the set of functions of $C_{r-1}([a,b])$ which are polynomials of degree at most r on $[x_j, x_{j+1}]$, for every $j \in \{0,\ldots,N-1\}$.*

We take analogous notations in the periodic setting. Let $N \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$. We consider the mesh

$$\Delta_\alpha \equiv \quad \alpha = x_0 < x_1 < \ldots < x_N = \alpha + 1$$

of the compact interval $[\alpha, \alpha+1]$.

Notation 2.7 *Let $\alpha \in \mathbb{R}$. We denote by $S^{0,\alpha}_{1-\mathrm{per}}(\mathbb{R},\Delta_\alpha)$ the set of all 1-periodic functions defined on $\mathbb{R}$ which are such that their restrictions to $[\alpha,\alpha+1]$ belong to $S^0([\alpha,\alpha+1],\Delta_\alpha)$. For $r \in \mathbb{N}_0$, we denote by $S^{r,\alpha}_{1-\mathrm{per}}(\mathbb{R},\Delta_\alpha)$ the set of all 1-periodic functions of $C_{r-1}(\mathbb{R})$ which are such that their restrictions to $[\alpha,\alpha+1]$ belong to $S^r([\alpha,\alpha+1],\Delta_\alpha)$.*

In the particular case of an uniform mesh with N mesh points in $[\alpha,\alpha+1[$, we define $\delta := \alpha N$ and

$$\Delta^{\mathrm{u}}_\delta \equiv \quad \frac{\delta}{N} = x_0 < x_1 < \ldots < x_N = \frac{\delta}{N} + 1,$$

with $x_k = \frac{k+\delta}{N}$, $\forall k \in \{0,\ldots,N\}$.

Notation 2.8 *For any $r \in \mathbb{N}$, any $\delta \in \mathbb{R}$ and any $N \in \mathbb{N}_0 \setminus \{1\}$, we let $S^{r,\delta,N}_{1-\text{per}}(\mathbb{R})$ be the space $S^{r,\frac{\delta}{N}}_{1-\text{per}}(\mathbb{R}, \Delta^{\text{u}}_{\delta})$.*

Let us introduce now some particular hierarchical spline spaces that will be useful as approximation spaces in the following.

Notation 2.9 *For any $m \in \mathbb{N}_0$ and $j \in \mathbb{N}$, we denote by $V^{(m)}_j$ the set of functions of $L^2(\mathbb{R})$ which are smoothest splines of degree $m-1$ with respect to the intervals $[2^{-j}k, 2^{-j}(k+1)[$, $k \in \mathbb{Z}$. If $\delta \in [0,1[$, we denote by $V^{(m)}_{j,\delta}$ the same set of splines but with respect to the intervals $[2^{-j}(k+\delta), 2^{-j}(k+\delta+1)[$, $k \in \mathbb{Z}$. The corresponding sets of 1-periodic splines are respectively denoted by $\mathcal{V}^{(m)}_j$, $\mathcal{V}^{(m)}_{j,\delta}$. In others words, we let $\mathcal{V}^{(m)}_j$ be the space $S^{m-1,0,2^j}_{1-\text{per}}(\mathbb{R})$ and $\mathcal{V}^{(m)}_{j,\delta}$ be the space $S^{m-1,\delta,2^j}_{1-\text{per}}(\mathbb{R})$.*

Proposition 2.10 *Let $s \in \mathbb{R}$, $m \in \mathbb{N}_0$, $j \in \mathbb{N}_0$ and $\delta \in [0,1[$. We have*

$$s < m - \frac{1}{2} \quad \Leftrightarrow \quad \mathcal{V}^{(m)}_{j,\delta} \subset H^s_{1-\text{per}}(\mathbb{R}).$$

PROOF. First of all, it is clear that $\mathcal{V}^{(m)}_{j,\delta} \in H^s_{1-\text{per}}(\mathbb{R})$ if and only if $\mathcal{V}^{(m)}_j \in H^s_{1-\text{per}}(\mathbb{R})$.

Suppose that the condition $s < m - \frac{1}{2}$ is satisfied. As it is proved in [5], the functions

$$\begin{aligned} g_0(x) &= 1, \\ g_\ell(x) &= \sum_{k\in\mathbb{Z}} \left(\frac{\ell}{\ell + k2^j}\right)^m e^{2i\pi(\ell+k2^j)x}, \quad -2^{j-1} < \ell \leq 2^{j-1},\ \ell \in \mathbb{Z}_0, \end{aligned}$$

form a basis of $\mathcal{V}^{(m)}_j$. We conclude by using the fact that for every $\ell \in \mathbb{N}$ such that $-2^{j-1} < \ell \leq 2^{j-1}$, the function g_ℓ belongs to $H^s_{1-\text{per}}(\mathbb{R})$ since

$$\sum_{k\in\mathbb{Z}} |\ell + k2^j|^{2s} \left(\frac{|\ell|}{|\ell + k2^j|}\right)^{2m} < +\infty.$$

Now let us assume that $\mathcal{V}^{(m)}_j \subset H^s_{1-\text{per}}(\mathbb{R})$. Then we have $s < m - \frac{1}{2}$ since otherwise we would have $\mathcal{V}^{(m)}_j \subset H^{m-\frac{1}{2}}_{1-\text{per}}(\mathbb{R})$. This would lead to an absurdity since the series

$$\sum_{k\in\mathbb{Z}} (1 + |1 + k2^j|)^{2m-1} \frac{1}{|1 + k2^j|^{2m}}$$

does not converge. ∎

From [5], we have the following result.

Proposition 2.11 *Let $m \in \mathbb{N}_0$, $j \in \mathbb{N}$ and f a 1-periodic function of $L^2_{\text{loc}}(\mathbb{R})$. The function f belongs to $\mathcal{V}^{(m)}_j$ if and only if*

$$k^m f_k = (k + 2^j)^m f_{k+2^j}$$

for all $k \in \mathbb{Z}$ where, as usual, $\forall k \in \mathbb{Z}$, the coefficient f_k denotes the k-th Fourier coefficient of f. ∎

We recall now two important approximation properties (see [39]). We need before to precise some notations and to give a definition.

Notation 2.12 *For $N \in \mathbb{N}_0$ and*

$$\Delta \equiv \quad 0 = x_0 < x_1 < \ldots < x_N = 1$$

a mesh on the compact interval $[0,1]$, we define the real numbers h_Δ and $\underline{h}_\Delta$ by

$$h_\Delta := \sup_{k \in \{1,\ldots,N\}} (x_k - x_{k-1}),$$
$$\underline{h}_\Delta := \inf_{k \in \{1,\ldots,N\}} (x_k - x_{k-1}).$$

Definition 2.13 *Let $\gamma \geq 1$. Such a mesh Δ is said to be γ-quasiuniform if the condition*

$$\frac{h_\Delta}{\underline{h}_\Delta} \leq \gamma$$

holds.

Proposition 2.14 (Inverse property for periodic splines) *Let $r \in \mathbb{N}$. For any numbers t, s and γ satisfying $t \leq s < r + \frac{1}{2}$ and $\gamma \geq 1$, there exists a constant $C > 0$ such that*

$$\|\varphi\|_{H^s_{1-\mathrm{per}}(\mathbb{R})} \leq C h_\Delta^{t-s} \|\varphi\|_{H^t_{1-\mathrm{per}}(\mathbb{R})}$$

for all $\varphi \in S^{r,0}_{1-\mathrm{per}}(\mathbb{R}, \Delta)$ with Δ a γ-quasiuniform mesh. ∎

Proposition 2.15 (Approximation property for periodic splines) *Let $r \in \mathbb{N}$. If the real numbers t, s are such that $t \leq s \leq r+1$ and $t < r + \frac{1}{2}$, then there exists a constant C independent of Δ such that*

$$\|u - P_{t,\Delta} u\|_{H^t_{1-\mathrm{per}}(\mathbb{R})} \leq C h_\Delta^{s-t} \|u\|_{H^s_{1-\mathrm{per}}(\mathbb{R})}, \quad \forall u \in H^s_{1-\mathrm{per}}(\mathbb{R}),$$

where $P_{t,\Delta}$ is the orthogonal projection onto $S^{r,0}_{1-\mathrm{per}}(\mathbb{R}, \Delta)$ in $H^t_{1-\mathrm{per}}(\mathbb{R})$. ∎

The next proposition will be useful many times in the following. The first part is simply the approximation property for splines applied to uniform dyadic meshes. The second part is established in [5].

Proposition 2.16 *Let $r \in \mathbb{N}$ and t, s be real numbers such that $0 \leq t \leq s \leq r+1$ and $t < r + \frac{1}{2}$. There is a constant $C > 0$ such that for every $f \in H^s_{1-\mathrm{per}}(\mathbb{R})$ and every $j \in \mathbb{N}$, there exists a spline function $S \in \mathcal{V}_j^{(r+1)}$ which satisfies*

$$\|f - S\|_{H^t_{1-\mathrm{per}}(\mathbb{R})} \leq C 2^{-(s-t)j} \|f\|_{H^s_{1-\mathrm{per}}(\mathbb{R})}.$$

Moreover, if $s < r+1$, then for every $f \in H^s_{1-\mathrm{per}}(\mathbb{R})$, we have

$$\inf_{S \in \mathcal{V}_j^{(r+1)}} \|f - S\|_{H^s_{1-\mathrm{per}}(\mathbb{R})} \to 0$$

if $j \to +\infty$. ∎

Corollary 2.17 *For every $\delta \in [0,1[$, every $r \in \mathbb{N}_0$ and every real number s such that $s < r$, the union $\cup_{j \in \mathbb{N}} \mathcal{V}_{j,\delta}^{(r)}$ is dense in $H^s_{1-\mathrm{per}}(\mathbb{R})$.* ∎

2.3 Chui-Wang wavelets ψ_m

Let m be fixed in $\mathbb{N}_0$.

2.3.1 Construction of the Chui-Wang wavelets

Let

$$N_m := \underbrace{\chi_{[0,1]} * \cdots * \chi_{[0,1]}}_{m \text{ factors}}$$

be the cardinal spline function. The classical Chui-Wang spline wavelet $\psi_m \in V_1^{(m)}$ is defined by

$$\widehat{\psi}_m(2\xi) := p_m(\xi)\widehat{N}_m(\xi) = p_m(\xi)e^{-im\frac{\xi}{2}}\left(\frac{\sin(\frac{\xi}{2})}{\frac{\xi}{2}}\right)^m$$

with

$$p_m(\xi) := e^{-i(m-1)\xi}\left(\frac{1-e^{-i\xi}}{2}\right)^m \omega_m(\xi+\pi)$$

and

$$\omega_m(\xi) := \sum_{k=-\infty}^{+\infty} |\widehat{N}_m(\xi+2k\pi)|^2 = \sum_{k=-m+1}^{m-1} e^{-ik\xi}N_{2m}(m+k).$$

Let $\psi_{m;j,k} := 2^{\frac{j}{2}}\psi_m(2^j\cdot -k)$ if $j,k\in\mathbb{Z}$.

Notation 2.18 *For every integer j, we denote by $W_j^{(m)}$ the orthogonal complement in $L^2(\mathbb{R})$ of $V_j^{(m)}$ in $V_{j+1}^{(m)}$.*

The following results can be found in [13].

Proposition 2.19 *The following properties hold*

1. *the functions $\psi_{m;0,k}$, $k\in\mathbb{Z}$, form a Riesz basis of $W_0^{(m)}$; hence, $\forall j\in\mathbb{Z}$, the functions $\psi_{m;j,k}$, $k\in\mathbb{Z}$, form a Riesz basis of $W_j^{(m)}$ with bounds independent of j;*

2. *the function ψ_m is compactly supported in $[0,2m-1]$ and has a symmetry (or antisymmetry) axis*

$$\psi_m(2m-1-x) = (-1)^m\psi_m(x), \quad \forall x\in\mathbb{R};$$

3. *we have orthogonality between levels*

$$\langle \psi_{m;j,k}\,,\, \psi_{m;j',k'}\rangle_{L^2(\mathbb{R})} = 0 \quad \text{if } j\neq j'.$$

∎

Corollary 2.20 *The functions $\psi_{m;j,k}(x)$, $j,k\in\mathbb{Z}$, form a Riesz basis of $L^2(\mathbb{R})$.* ∎

2.3.2 Periodization

Let

$$\Psi_{m;j} := 2^{\frac{j}{2}} \sum_{k=-\infty}^{+\infty} \psi_m(2^j(\cdot - k)), \quad j \in \mathbb{N}$$

and

$$\Psi_{m;j,k} := \Psi_{m;j}(\cdot - k2^{-j}), \quad j \in \mathbb{N},\ 0 \le k < 2^j.$$

Notation 2.21 *For every integer j, we denote by $\mathcal{W}_j^{(m)}$ the orthogonal complement in $L^2(]0,1[)$ of $\mathcal{V}_j^{(m)}$ in $\mathcal{V}_{j+1}^{(m)}$.*

Proposition 2.22 *We have the following properties*

1. *for every $j \in \mathbb{N}$, the functions $\Psi_{m;j,k}, 0 \le k < 2^j$, form a Riesz basis of $\mathcal{W}_j^{(m)}$ with bounds independent of j;*
2. *we have orthogonality between the levels*

$$\langle \Psi_{m;j,k}\,,\ \Psi_{m;j',k'} \rangle_{L^2(]0,1[)} = 0 \quad \text{if } j \neq j';$$

3. *the spline functions 1 and $\Psi_{m;j,k}$, $j \in \mathbb{N}$, $k \in \{0, \ldots, 2^j - 1\}$, form a Riesz basis of $L^2_{1-\text{per}}(\mathbb{R})$; moreover, after normalization of the constant function 1, the Riesz bounds in $L^2_{1-\text{per}}(\mathbb{R})$ are the same as the ones obtained for the functions $\psi_{m;j,k}$ ($j, k \in \mathbb{Z}$) in $L^2(\mathbb{R})$ (see Proposition 2.19).*

∎

2.3.3 Stability

The stability of Chui-Wang wavelets in Sobolev spaces has been studied in [36] and [18]. We give in [8] a simple proof in the periodic setting with the optimal indexes.

Proposition 2.23 *If the real number s satisfies $|s| < m - \frac{1}{2}$ then there are constants $c, C > 0$ such that*

$$c \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} |c_{j,k}|^2 \le \| \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} c_{j,k} 2^{-js} \Psi_{m;j,k} \|^2_{H^s_{1-\text{per}}(\mathbb{R})} \le C \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} |c_{j,k}|^2,$$

for every sequence $(c_{j,k})_{j \in \mathbb{N},\, k \in \{0, \ldots, 2^j-1\}}$ satisfying $\sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} |c_{j,k}|^2 < +\infty$.

PROOF. See [8]. ∎

Corollary 2.24 *If the real number s is such that $|s| < m - \frac{1}{2}$, then the functions 1 and $2^{-js}\Psi_{m;j,k}$ ($j \in \mathbb{N}$; $k \in \{0, \ldots, 2^j - 1\}$) form a Riesz basis of $H^s_{1-\text{per}}(\mathbb{R})$.*

PROOF. The Riesz condition is clearly given by Proposition 2.23 since we have

$$\|c_0 + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} c_{j,k} 2^{-js} \Psi_{m;j,k}\|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})} = |c_0|^2 + \|\sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} c_{j,k} 2^{-js} \Psi_{m;j,k}\|^2_{H^s_{1-\mathrm{per}}(\mathbb{R})}$$

for every c_0 and every sequence $(c_{j,k})_{j\in\mathbb{N},\, k\in\{0,\ldots,2^j-1\}}$ such that $\sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} |c_{j,k}|^2 < +\infty$.

For every $j \in \mathbb{N}$, the functions $\Psi_{m;j,k}$ $(k \in \{0,\ldots,2^j-1\})$ are 1-periodic spline functions of degree $m-1$ with respect to the mesh Δ_0^u with $N = 2^{j+1}$ mesh points. The space $\mathcal{V}_{j+1}^{(m)}$ of these spline functions has the dimension 2^{j+1} and we have exactly $1+1+2+\ldots+2^j = 2^{j+1}$ elements of this space among the Riesz family. So we can conclude since the union of these spaces is dense in $H^s_{1-\mathrm{per}}(\mathbb{R})$ (see Corollary 2.17). ∎

Remark 2.25 *The condition on the real number s given in Proposition 2.23 is optimal since $\{\Psi_{m;j} : j \in \mathbb{N}\} \subset H^s_{1-\mathrm{per}}(\mathbb{R})$ if and only if $s < m - \frac{1}{2}$.*

2.3.4 Some pictures

On $[0,1]$:

$\underline{m=1}$

$\Psi_{1;0}$:

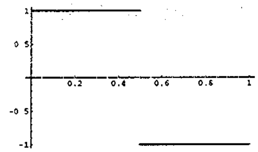


Fig 3

$$\Psi_{1;j}(x) = 2^{\frac{j}{2}} \Psi_{1;0}(2^j x) \chi_{[0,2^{-j}]}(x), \quad \forall j \geq 1$$

$\underline{m=2}$

$\Psi_{2;0}$, $\Psi_{2;1}$, $\Psi_{2;2}$:

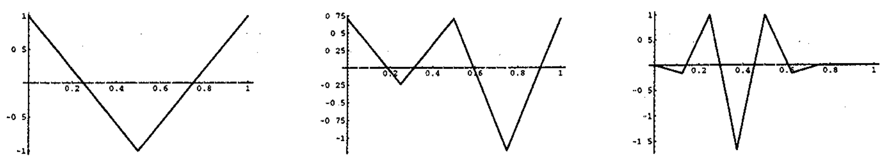


Fig 4

$$\Psi_{2;j}(x) = 2^{\frac{j-2}{2}}\Psi_{2;2}(2^{j-2}x)\chi_{[0,2^{-j+2}]}(x), \quad \forall j \geq 3$$

<u>$m = 3$</u>
$\Psi_{3;0},\ \Psi_{3;1},\ \Psi_{3;2},\ \Psi_{3;3}$:

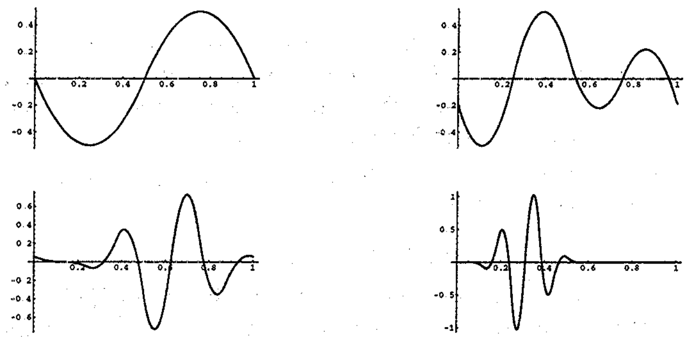


Fig. 5

$$\Psi_{3;j}(x) = 2^{\frac{j-3}{2}}\Psi_{3;3}(2^{j-3}x)\chi_{[0,2^{-j+3}]}(x), \quad \forall j \geq 4.$$

2.4 The functions θ_m

2.4.1 Construction of the functions $\psi_m^{(\ell)}$

Let m be a strictly positive integer and $\ell \in \{1, \ldots, m\}$.

Definition 2.26 *We define the function* $\psi_m^{(\ell)}$ *by*

$$\psi_m^{(\ell)}(x) := \frac{1}{(\ell-1)!}\int_{-\infty}^{x}(x-t)^{\ell-1}\psi_m(t)\,dt.$$

Proposition 2.27 *The function* $\psi_m^{(\ell)}$ *is the unique function of* $V_1^{(m+\ell)}$ *such that its* ℓ*-th derivative is the Chui-Wang wavelet* ψ_m *and which has the same support*

$$\mathrm{D}^\ell\psi_m^{(\ell)} = \psi_m, \quad \mathrm{supp}(\psi_m^{(\ell)}) = [0, 2m-1].$$

From the analogous properties on the Chui-Wang wavelet ψ_m*, the function* $\psi_m^{(\ell)}$ *has a symmetry (or antisymmetry) axis*

$$\psi_m^{(\ell)}(2m-1-x) = (-1)^{m+\ell}\psi_m^{(\ell)}(x), \quad \forall x \in \mathbb{R},$$

and has got $m-\ell$ *vanishing moments.* ∎

As for the Chui-Wang wavelets, let $\psi_{m;j,k}^{(\ell)} := 2^{\frac{j}{2}}\psi_m^{(\ell)}(2^j \cdot -k)$ if $j, k \in \mathbb{Z}$.

2.4.2 Functions θ_m

Definition 2.28 *For every $m \in \mathbb{N}_0$, we denote by θ_m the function $\psi_m^{(m)}$.*

Proposition 2.29 *For every $m \in \mathbb{N}_0$, the function θ_m vanishes at every integer*

$$\theta_m(p) = 0, \quad \forall p \in \mathbb{Z}.$$

PROOF. See [8]. ∎

From the next proposition (see [24]), we deduce that since, $\forall m \in \mathbb{N}_0$, the function θ_m belongs to the class $C_{2m-2}(\mathbb{R})$, is compactly supported, and has no vanishing moment, the function θ_m is not an orthogonal mother wavelet in $L^2(\mathbb{R})$. Let us first of all remind the definition of the 0-regularity.

Definition 2.30 *A function f of $L^2(\mathbb{R})$ is said to be 0-regular if for any $N \in \mathbb{N}$, there exists a constant C such that*

$$(1 + |x|)^N |f(x)| \leq C, \quad a.e.$$

Proposition 2.31 *If ψ is an orthonormal mother wavelet of $L^2(\mathbb{R})$ which belongs to the class $C_m(\mathbb{R})$ and is 0-regular, then the function ψ has $m+1$ vanishing moments.* ∎

2.4.3 Some pictures of the functions θ_m

θ_1, θ_2, θ_3 :

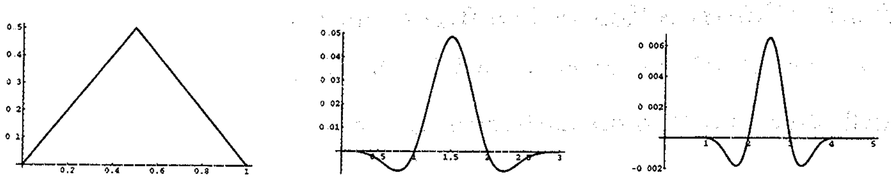


Fig. 6

θ_4 , θ_5 :

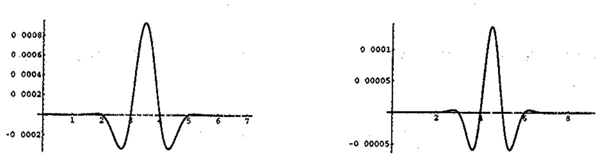


Fig. 7

2.4.4 Periodization, functions $\Psi^{(\ell)}_{m;j}$

Let $m \in \mathbb{N}_0$ and $\ell \in \{1, \ldots, m\}$. We define the functions $\Psi^{(\ell)}_{m;j}$ and $\Psi^{(\ell)}_{m;j,k}$ by

$$\Psi^{(\ell)}_{m;j} := 2^{\frac{j}{2}-\ell j} \sum_{k\in\mathbb{Z}} \psi^{(\ell)}_m(2^j(\cdot - k)), \quad j \in \mathbb{N},$$

and

$$\Psi^{(\ell)}_{m;j,k} := \Psi^{(\ell)}_{m;j}(\cdot - k2^{-j}), \quad j \in \mathbb{N},\ k \in \{0, \ldots, 2^j - 1\}.$$

We have chosen the factor $2^{-\ell j}$ in our periodization process in order to have $D^\ell \Psi^{(\ell)}_{m;j} = \Psi_{m;j}$, $\forall j \in \mathbb{N}$.

We get

$$\operatorname{supp}(\Psi^{(\ell)}_{m;j}) \subset \bigcup_{k\in\mathbb{Z}} [k, k + 2^{-j}(2m-1)].$$

In $[0, 1]$, this set is reduced to an interval with length $2^{-j}(2m-1)$ if j is larger than $J_m := \frac{\ln(2m-1)}{\ln(2)}$. For $j \geq J_m$, we have on $[0, 1]$

$$\Psi^{(\ell)}_{m;j}(x) = 2^{-\ell j+\frac{j}{2}} \psi^{(\ell)}_m(2^j x).$$

It follows that, $\forall j \geq J_m$, $\forall p \in \mathbb{N}$, we have

$$\Psi^{(\ell)}_{m;j+p}(x) = 2^{(-\ell+\frac{1}{2})p} \Psi^{(\ell)}_{m;j}(2^p x) \chi_{\cup_{k\in\mathbb{Z}}[k,k+2^{-p}]}(x), \quad \forall x \in \mathbb{R}.$$

For any $j \geq J_m$, the function $\Psi^{(\ell)}_{m;j}|_{[0,1]}$ reduced to its support is symmetric. Indeed, we have, $\forall x \in [0, 2^{-j}(2m-1)]$,

$$\Psi^{(\ell)}_{m;j}(2^{-j}(2m-1) - x) = 2^{-\ell j+\frac{j}{2}} \psi^{(\ell)}_m(2m - 1 - 2^j x) = 2^{-\ell j+\frac{j}{2}} \psi^{(\ell)}_m(2^j x) = \Psi^{(\ell)}_{m;j}(x).$$

2.4.5 Functions $\Theta_{m;j}$

Definition 2.32 *Let $m \in \mathbb{N}_0$. For every $j \in \mathbb{N}$ and any $k \in \{0, \ldots, 2^j - 1\}$, we denote respectively by $\Theta_{m;j}$ and $\Theta_{m;j,k}$ the functions $\Psi^{(m)}_{m;j}$ and $\Psi^{(m)}_{m;j,k}$.*

Using Proposition 2.29, we directly obtain the next result.

Proposition 2.33 *For every $m \in \mathbb{N}_0$, every $j \in \mathbb{N}$ and every $k \in \{0, \ldots, 2^j - 1\}$, the function $\Theta_{m;j,k}$ vanishes at 0.* ∎

Proposition 2.34 *Let $m \in \mathbb{N}_0$. The functions $\Theta_{m;j,k}$ also satisfy the orthogonality between the levels j for the scalar product $\langle \cdot, \cdot \rangle_m$, i.e.*

$$\langle \Theta_{m;j,k}, \Theta_{m;j',k'} \rangle_m = 0$$

for any $j, j' \in \mathbb{N}$ such that $j \neq j'$ and any $k \in \{0, \ldots, 2^j - 1\}$, $k' \in \{0, \ldots, 2^{j'} - 1\}$. ∎

Proposition 2.35 *For every $m \in \mathbb{N}_0$, there is a constant $C_1^{(m)}$ such that*

$$\sup_{j\in\mathbb{N}} \sup_{k\in\{0,\ldots,2^j-1\}} \sup_{x\in\mathbb{R}} 2^{(m-\frac{1}{2})j}|\Theta_{m;j,k}(x)| \leq C_1^{(m)}.$$

PROOF. For $j \geq J_m$, we have

$$\begin{aligned}\sup_{x\in\mathbb{R}} 2^{(m-\frac{1}{2})j}|\Theta_{m;j}(x)| &= \sup_{x\in[0,1]} 2^{(m-\frac{1}{2})j}|\Theta_{m;j}(x)| \\ &= \sup_{x\in[0,1]} |\theta_m(2^j x)| \leq \sup_{y\in\mathbb{R}} |\theta_m(y)| < +\infty.\end{aligned}$$

∎

2.4.6 Stability

Proposition 2.36 *For any $m \in \mathbb{N}_0$, any $\ell \in \{1,\ldots,m-1\}$ and any real number s such that $\frac{1}{2}+\ell-m < s < \ell+m-\frac{1}{2}$, the functions 1 and $2^{j(\ell-s)}\Psi^{(\ell)}_{m;j,k}$, $j \in \mathbb{N}$, $k \in \{0,\ldots,2^j-1\}$, form a Riesz basis of $H^s_{1-\text{per}}(\mathbb{R})$.*

PROOF. By construction, we have $(\Psi^{(\ell)}_{m;j,k})_p = (2i\pi p)^{-\ell}(\Psi_{m;j,k})_p$, $\forall j \in \mathbb{N}$, $k \in \{0,\ldots 2^j-1\}$, $p \in \mathbb{Z}_0$ (here the index p means that we take the p-th Fourier coefficient). Since $\ell \in \{1,\ldots,m-1\}$, the function $\psi^{(\ell)}_m$ has at least one vanishing moment; hence we have

$$\int_0^1 \Psi^{(\ell)}_{m;j,k}(x)\,dx = 0, \quad \forall j \in \mathbb{N},\ \forall k \in \{0,\ldots,2^j-1\}$$

and we get

$$\begin{aligned}&\left\| c_0 + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} c_{j,k}2^{j(\ell-s)}\Psi^{(\ell)}_{m;j,k}\right\|^2_{H^s_{1-\text{per}}(\mathbb{R})} \\ &= |c_0|^2 + (2\pi)^{-2\ell}\sum_{p\in\mathbb{Z}_0}|p|^{2(s-\ell)}\left|\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} c_{j,k}2^{j(\ell-s)}(\Psi_{m;j,k})_p\right|^2 \\ &= |c_0|^2 + (2\pi)^{-2\ell}\left\|\sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} c_{j,k}2^{j(\ell-s)}\Psi_{m;j,k}\right\|^2_{H^{s-\ell}_{1-\text{per}}(\mathbb{R})}\end{aligned}$$

We conclude using on one hand Proposition 2.23 and on the other hand a density argument since, for every $j \in \mathbb{N}$, the functions $\Psi^{(\ell)}_{m;j,k}$ ($k \in \{0,\ldots,2^j-1\}$) belong to $\mathcal{V}^{(m+\ell)}_{j+1}$. ∎

We get the same property for $\ell = m$ but the proof is technically different because the function θ_m has no more vanishing moment. To get the Riesz condition, we use in this case and for $s > \frac{1}{2}$ the scalar product $\langle\cdot,\cdot\rangle_s$ defined in the beginning of this chapter.

Proposition 2.37 *For any $m \in \mathbb{N}_0$ and any real number s such that $\frac{1}{2} < s < 2m - \frac{1}{2}$, the functions* 1 *and* $2^{j(m-s)}\Theta_{m;j,k}$, $j \in \mathbb{N}$, $k \in \{0, \ldots, 2^j - 1\}$, *form a Riesz basis of* $H^s_{1-\mathrm{per}}(\mathbb{R})$.

PROOF. See [8]. ∎

In the proof of Proposition 2.37, we got also the following multiscale result.

Proposition 2.38 *For any $m \in \mathbb{N}_0$, any $s \in \mathbb{R}$ such that $\frac{1}{2} < s < 2m - \frac{1}{2}$ and any $j \in \mathbb{N}_0$, the functions* 1 *and* $2^{\ell(m-s)}\Theta_{m;\ell,k}$, $\ell \in \{0, \ldots, j-1\}$, $k \in \{0, \ldots, 2^\ell - 1\}$, *form a Riesz basis of* $\mathcal{V}_j^{(2m)}$. *Moreover, the Riesz bounds are independent of j.* ∎

2.4.7 $\Theta_{m;j}$ pictures

On $[0,1]$:

<u>$m = 1$</u>

$$\Theta_{1;0} = \theta_1$$
$$\Theta_{1;j}(x) = 2^{\frac{-j}{2}}\Theta_{1;0}(2^j x)\chi_{[0,2^{-j}]}(x), \quad \forall j \geq 1$$

<u>$m = 2$</u>

$\Theta_{2;0}$, $\Theta_{2;1}$, $\Theta_{2;2}$:

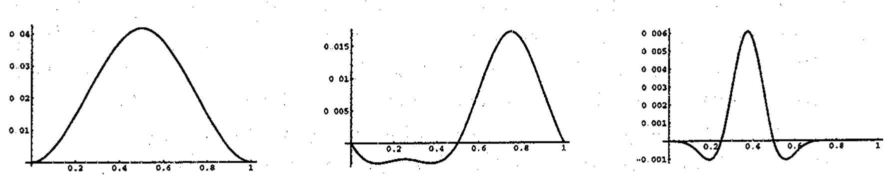


Fig 8

$$\Theta_{2;j}(x) = 2^{\frac{-3(j-2)}{2}}\Theta_{2;2}(2^{j-2}x)\chi_{[0,2^{-j+2}]}(x), \quad \forall j \geq 3$$

<u>$m = 3$</u>

$\Theta_{3;0}$, $\Theta_{3;1}$, $\Theta_{3;2}$, $\Theta_{3;3}$:

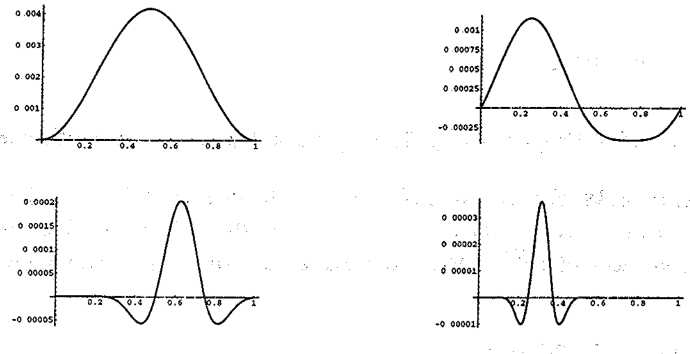


Fig 9

$$\Theta_{3;j}(x) = 2^{\frac{-5(j-3)}{2}} \Theta_{3;3}(2^{j-3}x)\chi_{[0,2^{-j+3}]}(x), \quad \forall j \geq 4.$$

2.5 Dual Riesz basis

2.5.1 Problem

Let $m \in \mathbb{N}_0$, $\ell \in \{0, \ldots, m\}$ and $s \in \mathbb{R}$ such that $\frac{1}{2} + \ell - m < s < \ell + m - \frac{1}{2}$. We have proved (see Corollary 2.24 and Propositions 2.36, 2.37) that the family

$$\{u_p : p \in \mathbb{N}\} = \{1\} \cup \{2^{j(\ell-s)}\Psi^{(\ell)}_{m;j,k} : j \in \mathbb{N},\, k \in \{0, \ldots, 2^j - 1\}\}$$

is a Riesz basis of the Hilbert space $H^s_{1-\mathrm{per}}(\mathbb{R})$ (here $\Psi^{(0)}_{m;j,k} := \Psi_{m;j,k}$). So, we know that there exists a unique Riesz basis $\{v_q : q \in \mathbb{N}\}$ of $H^s_{1-\mathrm{per}}(\mathbb{R})$ (resp. $H^m_{1-\mathrm{per}}(\mathbb{R})$) such that $\langle u_p\,,\, v_q\rangle_{H^s_{1-\mathrm{per}}(\mathbb{R})} = \delta_{p,q}$ (resp. $\langle u_p\,,\, v_q\rangle_m = \delta_{p,q}$), $\forall p, q \in \mathbb{N}$. We call it the *dual Riesz basis* of the Riesz basis $\{u_p : p \in \mathbb{N}\}$ for the scalar product $\langle \cdot\,,\,\cdot\rangle_{H^s_{1-\mathrm{per}}(\mathbb{R})}$ (resp. for the scalar product $\langle \cdot\,,\,\cdot\rangle_m$). This is the unique family $\{v_q : q \in \mathbb{N}\}$ of functions of $H^s_{1-\mathrm{per}}(\mathbb{R})$ (resp. $H^m_{1-\mathrm{per}}(\mathbb{R})$) such that $\langle u_p\,,\, v_q\rangle_{H^s_{1-\mathrm{per}}(\mathbb{R})} = \delta_{p,q}$ (resp. $\langle u_p\,,\, v_q\rangle_m = \delta_{p,q}$), $\forall p, q \in \mathbb{N}$.

It would be interesting to know if these dual Riesz bases of the Riesz basis $\{u_p : p \in \mathbb{N}\}$ have also a multiscale frame, i.e. if we have

$$\{v_q : q \in \mathbb{N}\} = \{1\} \cup \{c_j \widetilde{\Psi}^{(\ell)}_{m;j,k} : j \in \mathbb{N},\, k \in \{0, \ldots, 2^j - 1\}\}$$

with

$$\begin{aligned} \widetilde{\Psi}^{(\ell)}_{m;j} &= 2^{-mj+\frac{j}{2}} \sum_{k\in\mathbb{Z}} \widetilde{\psi}^{(\ell)}_m(2^j(\cdot - k)), \quad \forall j \in \mathbb{N}, \\ \widetilde{\Psi}^{(\ell)}_{m;j,k} &= \widetilde{\Psi}^{(\ell)}_{m;j}(\cdot - k2^{-j}), \quad \forall j \in \mathbb{N},\, \forall k \in \{0, \ldots, 2^j - 1\} \end{aligned}$$

for a function $\widetilde{\psi}_m^{(\ell)}$ of $V_1^{(m+\ell)}$ and some constants c_j which can depend on m and s.

For $\ell = m$, the dual Riesz basis for the scalar product $\langle \cdot , \cdot \rangle_{H^s_{1-\mathrm{per}}(\mathbb{R})}$ can not be written as above. Indeed, we have

$$\langle \Theta_{m;j,k} , 1 \rangle_{H^s_{1-\mathrm{per}}(\mathbb{R})} = (\Theta_{m;j})_0 \neq 0, \quad \forall j \in \mathbb{N},\ \forall k \in \{0, \dots, 2^j - 1\}.$$

For $\ell \in \{0, \dots, m-1\}$, the multiscale form is obtained for this scalar product and $s = \ell$. We present this construction below.

For $\ell = m$, the scalar product $\langle \cdot , \cdot \rangle_s$ seems better convenient to give such a multiscale result. We obtain good results for $s = m$. This construction is also presented below.

Both of our constructions are carried out using primitives of the classical dual Chui-Wang wavelets. Let us remind the basic properties of these dual Chui-Wang wavelets.

2.5.2 Dual Chui-Wang Riesz basis in $L^2(\mathbb{R})$

Let $m \in \mathbb{N}_0$.

Proposition 2.39 *If ψ_m is the Chui-Wang wavelet, the function $\widetilde{\psi}_m$ defined by*

$$\widehat{\widetilde{\psi}}_m := \frac{\widehat{\psi}_m}{\sum_{k\in\mathbb{Z}} |\widehat{\psi}_m(\cdot + 2k\pi)|^2}$$

satisfy the following properties

- *the family $\{\widetilde{\psi}_m(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz basis of $W_0^{(m)}$ (see Notation 2.18);*
- *the family $\{\widetilde{\psi}_{m;j,k} := 2^{\frac{j}{2}}\widetilde{\psi}_m(2^j \cdot - k) : j, k \in \mathbb{Z}\}$ is a Riesz basis of $L^2(\mathbb{R})$;*
- $\langle \psi_m , \widetilde{\psi}_m(\cdot - k) \rangle_{L^2(\mathbb{R})} = \delta_{k,0}$, $\forall k \in \mathbb{Z}$;
- $\langle \widetilde{\psi}_{m;j,k} , \widetilde{\psi}_{m;j',k'} \rangle_{L^2(\mathbb{R})} = 0$, $\forall j, j', k, k' \in \mathbb{Z}$ *such that* $j \neq j'$;
- *$\widetilde{\psi}_m$ is exponentially fast decaying, i.e. there exist constants $\beta > 0$ and $F > 0$ such that $|\widetilde{\psi}_m(x)| \leq F e^{-\beta|x|}$, $\forall x \in \mathbb{R}$.*

∎

Corollary 2.40 *We have*

$$\widetilde{\psi}_m = \sum_{k\in\mathbb{Z}} c_k \psi_m(\cdot - k)$$

with

$$c_k := \langle \widetilde{\psi}_m , \widetilde{\psi}_m(\cdot - k) \rangle_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ik\xi}}{\sum_{\ell\in\mathbb{Z}} |\widehat{\psi}_m(\xi + 2\ell\pi)|^2} d\xi, \quad \forall k \in \mathbb{Z}$$

and this sequence $(c_k)_{k\in\mathbb{Z}}$ is exponentially fast decaying, i.e. there exist constants $\alpha > 0$ and $C > 0$ such that

$$|c_k| \leq C\, e^{-\alpha|k|}, \quad \forall k \in \mathbb{Z}.$$

PROOF. The behavior of the sequence $(c_k)_{k\in\mathbb{Z}}$ can be obtained from the last item of Proposition 2.39. Indeed, with the notations of this item, we have

$$|\langle \widetilde{\psi}_m\,, \widetilde{\psi}_m(\cdot - k)\rangle_{L^2(\mathbb{R})}| \leq F^2\, e^{-\beta|k|}\left(\frac{1}{\beta} + |k|\right), \quad \forall k \in \mathbb{Z}.$$

∎

2.5.3 Functions $\widetilde{\psi}_m^{(\ell)}$

Let $m \in \mathbb{N}_0$ and $\ell \in \{1, \ldots, m\}$.

Definition 2.41 *The function $\widetilde{\psi}_m^{(\ell)}$ is defined from the dual Chui-Wang wavelet $\widetilde{\psi}_m$ by*

$$\widetilde{\psi}_m^{(\ell)}(x) := \frac{1}{(\ell-1)!}\int_{-\infty}^{x} (x-t)^{\ell-1}\, \widetilde{\psi}_m(t)\, dt, \quad x \in \mathbb{R}.$$

We denote by $\widetilde{\theta}_m$ the function $\widetilde{\psi}_m^{(m)}$.

Proposition 2.42 *The function $\widetilde{\psi}_m^{(\ell)}$ belongs to $C_{m+\ell-2}(\mathbb{R})$ and satisfies $\mathrm{D}^\ell \widetilde{\psi}_m^{(\ell)} = \widetilde{\psi}_m$ on $\mathbb{R}$.* ∎

Proposition 2.43 *The function $\widetilde{\psi}_m^{(\ell)}$ is exponentially fast decaying, i.e. there are constants $\beta' > 0$ and $F' > 0$ such that*

$$|\widetilde{\psi}_m^{(\ell)}(x)| \leq F' e^{-\beta'|x|}, \quad \forall x \in \mathbb{R}.$$

PROOF. We have

$$\widehat{\psi}_m(2\xi) = 2^{-m} e^{-i(m-1)\xi}\frac{(1-e^{-i\xi})^{2m}}{(i\xi)^m}\omega_m(\xi+\pi)$$

and

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|\widehat{\psi}_m(2\xi+2k\pi)|^2 &= \left(\sin^{2m}(\frac{\xi}{2})\,\omega_m(\xi+\pi) + \cos^{2m}(\frac{\xi}{2})\,\omega_m(\xi)\right)\omega_m(\xi)\,\omega_m(\xi+\pi)\\ &= \omega_m(2\xi)\,\omega_m(\xi)\,\omega_m(\xi+\pi).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\widehat{\widetilde{\psi}}_m^{(\ell)}(2\xi) &= \frac{\widehat{\psi}_m(2\xi)}{(2i\xi)^\ell \sum_{k\in\mathbb{Z}}|\widehat{\psi}_m(2\xi+2k\pi)|^2}\\ &= 2^{-(\ell+m)} e^{-i(m-1)\xi}\frac{(1-e^{-i\xi})^{2m}}{(i\xi)^{\ell+m}}\frac{1}{\omega_m(2\xi)\,\omega_m(\xi)}\\ &= 2^{-(\ell+m)} e^{-4i(m-1)\xi}\frac{(1-e^{-i\xi})^{2m}}{(i\xi)^{\ell+m}}\frac{1}{P(e^{-2i\xi})\,P(e^{-i\xi})}\end{aligned}$$

where we have defined the polynomial P by

$$P(z) := \sum_{k=-m+1}^{m-1} z^{m-1+k} N_{2m}(m+k).$$

Since

$$P(e^{-i\alpha}) = e^{-i(m-1)\alpha}\omega_m(\alpha), \quad \forall \alpha \in \mathbb{R},$$

this polynomial P has no zero of modulus 1 and we can define a_0 as the zero of P which has the largest modulus strictly smaller than 1. It is clear that $a_0 = \frac{1}{a_1}$ where a_1 is the zero of P which has the smallest modulus strictly larger than 1. Let $s_0 := -\ln(|a_0|)$ and $r_1 \in \mathbb{R}$ such that $0 < r_1 < s_0$. We get constants $C^{(1)}$ and $C^{(2)}$ such that

$$|\widehat{\widetilde{\psi}}_m^{(\ell)}(2(\xi + i\eta))| \leq \begin{cases} C^{(1)} & \text{if } |\eta| \leq \frac{r_1}{2},\ |\xi| \leq s_0 \\ \frac{C^{(2)}}{(|\xi| - \frac{r_1}{2})^{(m+\ell)}} & \text{if } |\eta| \leq \frac{r_1}{2},\ |\xi| \geq s_0. \end{cases}$$

Let $x \geq 0$. Integrating the function

$$f(z) := e^{ixz}\widehat{\widetilde{\psi}}_m^{(\ell)}(2z) \in \mathcal{O}(\{\xi + i\eta : \xi \in \mathbb{R},\ \eta \in]-\frac{s_0}{2}, \frac{s_0}{2}[\})$$

on the complex rectangular boundary defined by the vertices $(R, 0)$, $(R, \frac{r_1}{2})$, $(-R, \frac{r_1}{2})$, $(-R, 0)$, we get

$$\left|\widetilde{\psi}_m^{(\ell)}(x)\right| = \frac{1}{\pi}e^{-xr_1}\left|\int_{\mathbb{R}} e^{2ix\xi}\widehat{\widetilde{\psi}}_m^{(\ell)}(2(\xi + i\frac{r_1}{2}))d\xi\right| \leq F^{(1)}e^{-r_1|x|}.$$

Now, if $x < 0$, we integrate this function f on the rectangular boundary with vertices $(R, 0)$, $(R, -\frac{r_1}{2})$, $(-R, -\frac{r_1}{2})$, $(-R, 0)$ and we get

$$\left|\widetilde{\psi}_m^{(\ell)}(x)\right| = \frac{1}{\pi}e^{xr_1}\left|\int_{\mathbb{R}} e^{2ix\xi}\widehat{\widetilde{\psi}}_m^{(\ell)}(2(\xi - i\frac{r_1}{2}))d\xi\right| \leq F^{(2)}e^{-r_1|x|}.$$

This proves the proposition. ∎

Proposition 2.44 *The function $\widetilde{\psi}_m^{(\ell)}$ satisfies*

$$\widetilde{\psi}_m^{(\ell)} = \sum_{k \in \mathbb{Z}} c_k \psi_m^{(\ell)}(\cdot - k).$$

This implies $\widetilde{\psi}_m^{(\ell)} \in V_1^{(m+\ell)}$.

PROOF. We just have to remark that the functions $\widetilde{\psi}_m^{(\ell)}$ and $\sum_{k\in\mathbb{Z}} c_k \psi_m^{(\ell)}(\cdot - k)$ have the same regularity, the same ℓ-th derivative and the same behavior in $+\infty$. ∎

Corollary 2.45 *The function $\widetilde{\theta}_m$ vanishes at every integer*

$$\widetilde{\theta}_m(p) = 0, \quad \forall p \in \mathbb{Z}.$$

∎

Now we can apply the periodization process to these functions $\widetilde{\psi}_m^{(\ell)}$ ($\ell \in \{0, \dots, m\}$, $\widetilde{\psi}_m^{(0)} := \widetilde{\psi}_m$). We define, for every $j \in \mathbb{N}$, the functions $\widetilde{\Psi}_{m;j}^{(\ell)}$ and $\widetilde{\Theta}_{m;j}$ by

$$\widetilde{\Psi}_{m;j}^{(\ell)} := 2^{-\ell j+\frac{j}{2}} \sum_{p \in \mathbb{Z}} \widetilde{\psi}_m^{(\ell)}(2^j(\cdot - p)) = \sum_{k \in \mathbb{Z}} c_k \Psi_{m;j}^{(\ell)}(\cdot - k2^{-j}); \quad \widetilde{\Theta}_{m;j} := \widetilde{\Psi}_{m;j}^{(m)}$$

and, for every $j \in \mathbb{N}$, $k \in \{0, \dots, 2^j - 1\}$, the functions $\widetilde{\Psi}_{m;j,k}^{(\ell)}$ and $\widetilde{\Theta}_{m;j,k}$ by

$$\widetilde{\Psi}_{m;j,k}^{(\ell)} := \widetilde{\Psi}_{m;j}^{(\ell)}(\cdot - k2^{-j}); \quad \widetilde{\Theta}_{m;j,k} := \widetilde{\Psi}_{m;j,k}^{(m)}.$$

For every $j \in \mathbb{N}$, the factor $2^{-\ell j}$ in the expression of $\widetilde{\Psi}_{m;j}^{(\ell)}$ is chosen again in such a way that $\widetilde{\Psi}_{m;j}$ is the ℓ-th derivative of $\widetilde{\Psi}_{m;j}^{(\ell)}$.

With these functions $\widetilde{\Psi}_{m;j,k} = \widetilde{\Psi}_{m;j,k}^{(0)}$, we get

$$\langle \Psi_{m;j,k}\,,\ \widetilde{\Psi}_{m;j',k'} \rangle_{L^2(]0,1[)} = \delta_{j,j'}\delta_{k,k'}, \quad \forall j, j' \in \mathbb{N},\ k \in \{0, \dots, 2^j-1\},\ k' \in \{0, \dots, 2^{j'}-1\};$$

and

$$\langle \widetilde{\Psi}_{m;j,k}\,,\ \widetilde{\Psi}_{m;j',k'} \rangle_{L^2(]0,1[)} = 0, \quad \forall j, j' \in \mathbb{N} \text{ s.t. } j \neq j',\ k \in \{0, \dots, 2^j - 1\},\ k' \in \{0, \dots, 2^{j'} - 1\}.$$

Using Corollary 2.45, we obtain the following result.

Proposition 2.46 *For every $j \in \mathbb{N}$ and every $k \in \{0, \dots, 2^j-1\}$, the function $\widetilde{\Theta}_{m;j,k}$ vanishes at 0.* ∎

2.5.4 Dual basis for the scalar product $\langle \cdot\,,\ \cdot \rangle_{H^s_{1-\text{per}}(\mathbb{R})}$

Proposition 2.47 *Let $\ell \in \{0, \dots, m-1\}$. The dual Riesz basis of the basis*

$$\{1\} \cup \{\Psi_{m;j,k}^{(\ell)} : j \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\}\}$$

for the scalar product $\langle \cdot\,,\ \cdot \rangle_{H^\ell_{1-\text{per}}(\mathbb{R})}$ is the basis

$$\{1\} \cup \{(2\pi)^{2\ell} \widetilde{\Psi}_{m;j,k}^{(\ell)} : j \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\}\}.$$

Proof. Since $(\Psi_{m;j,k}^{(\ell)})_0 = 0$, $\forall j \in \mathbb{N}$, $\forall k \in \{0, \dots, 2^j - 1\}$, we get

$$\begin{aligned}
\langle \Psi_{m;j,k}^{(\ell)}\,,\ \widetilde{\Psi}_{m;j',k'}^{(\ell)} \rangle_{H^\ell_{1-\text{per}}(\mathbb{R})} &= (2\pi)^{-2\ell} \delta_{j,j'}\delta_{k,k'}, \\
&\qquad \forall j, j' \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\},\ k' \in \{0, \dots, 2^{j'} - 1\}; \\
\langle \Psi_{m;j,k}^{(\ell)}\,,\ 1 \rangle_{H^\ell_{1-\text{per}}(\mathbb{R})} &= 0, \quad \forall j \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\}; \\
\langle \widetilde{\Psi}_{m;j,k}^{(\ell)}\,,\ 1 \rangle_{H^\ell_{1-\text{per}}(\mathbb{R})} &= 0, \quad \forall j \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\}.
\end{aligned}$$

∎

2.5.5 Dual basis for the scalar product $\langle \cdot\,, \cdot \rangle_s$

Proposition 2.48 *The dual Riesz basis of the basis*

$$\{1\} \cup \{\Theta_{m;j,k} : j \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\}\}$$

for the scalar product $\langle \cdot\,, \cdot \rangle_m$ is the basis

$$\{1\} \cup \{(2\pi)^{2m} \widetilde{\Theta}_{m;j,k} : j \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\}\}.$$

PROOF. This is clear because, using Propositions 2.33 and 2.46, we get

$$\begin{aligned}
\langle \Theta_{m;j,k}\,, \widetilde{\Theta}_{m;j',k'} \rangle_m &= (2\pi)^{-2m} \delta_{j,j'} \delta_{k,k'}, \\
&\qquad \forall j, j' \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\},\ k' \in \{0, \dots, 2^{j'} - 1\}; \\
\langle \Theta_{m;j,k}\,, 1 \rangle_m &= 0, \quad \forall j \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\}; \\
\langle \widetilde{\Theta}_{m;j,k}\,, 1 \rangle_m &= 0, \quad \forall j \in \mathbb{N},\ k \in \{0, \dots, 2^j - 1\}.
\end{aligned}$$

∎

Remark 2.49 *Of course, we also have*

$$\langle \widetilde{\Theta}_{m;j,k}\,, \widetilde{\Theta}_{m;j',k'} \rangle_m = 0, \quad \forall j, j' \in \mathbb{N} \text{ s.t. } j \neq j',\ k \in \{0, \dots, 2^j - 1\},\ k' \in \{0, \dots, 2^{j'} - 1\}.$$

2.5.6 $\widetilde{\Theta}_{m;j}$ pictures

On $[0, 1]$:

<u>$m = 1$</u>

$$\widetilde{\Theta}_{1;0} = \Theta_{1,0} = \theta_1$$

<u>$m = 2$</u>

$$\widetilde{\Theta}_{2;0},\ \widetilde{\Theta}_{2;1},\ \widetilde{\Theta}_{2;2},\ \widetilde{\Theta}_{2;3} :$$

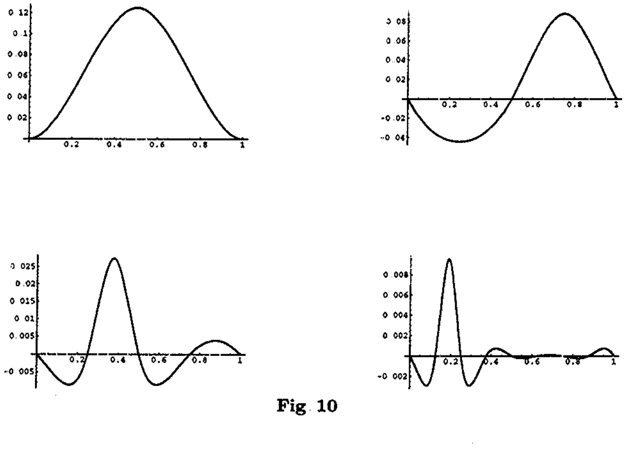


Fig. 10

$\underline{m = 3}$

$\widetilde{\Theta}_{3;0},\ \widetilde{\Theta}_{3;1},\ \widetilde{\Theta}_{3;2},\ \widetilde{\Theta}_{3;3}$:

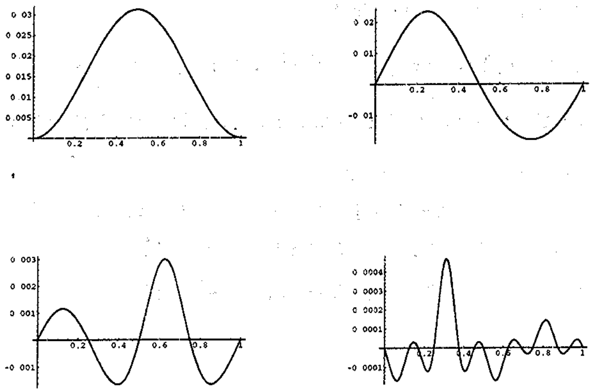


Fig. 11

Chapter 3

Collocation with spline wavelets

Let Ω be a smooth bounded and connected open subset of $\mathbb{R}^2$ whose boundary $\Gamma = \partial\Omega$ is a connected C_∞ Jordan curve. By "smooth", we mean that for every $x_0 \in \Gamma$, there exist an open neighborhood V_{x_0} of x_0, an open neighborhood U_{x_0} of 0 in $\mathbb{R}^2$ and $u(x)$ a C_∞ change of coordinates between V_{x_0} et U_{x_0} such that

- $u(x_0) = 0$;
- $u(V_{x_0} \cap \Omega) = \{u \in U_{x_0} : \ u_2 > 0\}$;
- $u(V_{x_0} \cap \Gamma) = \{u \in U_{x_0} : \ u_2 = 0\}$.

We also assume that the boundary Γ admits a Jordan parametrization of class C_∞

$$x = \gamma(t), \quad t \in [0,1],$$

with $\mathrm{D}_t\gamma(t) \neq 0, \ \forall t \in [0,1]$.

We define the operator K_γ acting on functions on Γ by

$$K_\gamma : \ v \mapsto v(\gamma(\cdot \bmod (1))).$$

In a natural way, we will often identify a function f defined on Γ with the function $F := K_\gamma f$.

With the parametrization γ, we can define Sobolev spaces on the boundary Γ (see [38]).

Definition 3.1 *For every $s \in \mathbb{R}$, we define $H^s(\Gamma)$ as the space of all distributions u on Γ such that $K_\gamma u \in H^s_{1-\mathrm{per}}(\mathbb{R})$ endowed with the norm*

$$\|u\|_{H^s(\Gamma)} = \|K_\gamma u\|_{H^s_{1-\mathrm{per}}(\mathbb{R})}.$$

From Definition 3.1, K_γ is an isometric isomorphism from $H^s(\Gamma)$ onto $H^s_{1-\mathrm{per}}(\mathbb{R})$, for every $s \in \mathbb{R}$.

Remark 3.2 *As it is proved in [28], the previous definition does not depend on the choice of a parametrization of the boundary as described above.*

We start this third chapter by a presentation of the classical theory of pseudodifferential operators on Γ.

3.1 Pseudodifferential operators on the boundary of a smooth open set of $\mathbb{R}^2$

Let $n \in \mathbb{N}_0$. To introduce pseudodifferential operators on $\mathbb{R}^n$ or on an open subset of $\mathbb{R}^n$, we follow the definitions of [2].

3.1.1 Pseudodifferential operators on $\mathbb{R}^n$

Definition 3.3 *Let $m \in \mathbb{R}$. We denote by $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ the set of functions $a \in C_\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that, $\forall \alpha, \beta \in \mathbb{N}^n$, there is a constant $C_{\alpha,\beta}$ such that*

$$|\mathrm{D}_x^\alpha \mathrm{D}_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \quad \forall x, \xi \in \mathbb{R}^n.$$

Such a function a is called a symbol of order m.

Notation 3.4 *We take the following notations*

$$S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n) := \cap_{m\in\mathbb{R}} S^m(\mathbb{R}^n \times \mathbb{R}^n), \qquad S^{\infty}(\mathbb{R}^n \times \mathbb{R}^n) := \cup_{m\in\mathbb{R}} S^m(\mathbb{R}^n \times \mathbb{R}^n).$$

Definition 3.5 *Let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. We define the pseudodifferential operator of symbol a and order m as follows*

$$\mathrm{Op}(a) = a(x, \mathrm{D}) : S(\mathbb{R}^n) \to S(\mathbb{R}^n) : \ u \mapsto (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi) d\xi.$$

From [2], we have the following propositions.

Proposition 3.6 *If $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, then the operator $\mathrm{Op}(a)$ can be extended to an operator from $S'(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$.* ∎

Proposition 3.7 *If $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ with $m < -n$, then $\mathrm{Op}(a)$ admits the kernel*

$$K(x,y) := (2\pi)^{-n} \mathcal{F}^{-}_{\xi \to y-x} a(x,\xi)$$

and $K \in C_\infty(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\})$. ∎

3.1.2 Pseudodifferential operators on an open subset of $\mathbb{R}^n$

Definition 3.8 *Let $m \in \mathbb{R}$ and Ω be an open subset of $\mathbb{R}^n$. We denote by $S^m_{\mathrm{loc}}(\Omega \times \mathbb{R}^n)$ the set of all functions a of $C_\infty(\Omega \times \mathbb{R}^n)$ such that $\varphi a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ for every $\varphi \in D(\Omega)$.*

Notation 3.9 *We take the notations*

$$S^{-\infty}_{\mathrm{loc}}(\Omega \times \mathbb{R}^n) := \cap_{m\in\mathbb{R}} S^m_{\mathrm{loc}}(\Omega \times \mathbb{R}^n), \quad S^{\infty}_{\mathrm{loc}}(\Omega \times \mathbb{R}^n) := \cup_{m\in\mathbb{R}} S^m_{\mathrm{loc}}(\Omega \times \mathbb{R}^n).$$

We can find the following useful results in [2] and [5].

Proposition 3.10 *If $a \in S^m_{\text{loc}}(\Omega \times \mathbb{R}^n)$, then the operator $\text{Op}(a) = a(x, \text{D})$ defined again by*

$$\text{Op}(a)u = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\,\xi} a(x,\xi)\widehat{u}(\xi)d\xi$$

is defined from $S'(\mathbb{R}^n)$ to $D'(\Omega)$. ∎

Proposition 3.11 *Let $A : D(\Omega) \to C_\infty(\Omega)$ a continuous linear operator such that*

$$\varphi A \psi \in \text{Op}(S^m(\mathbb{R}^n \times \mathbb{R}^n)), \quad \forall \varphi, \psi \in D(\Omega).$$

Then there exists $a' \in S^m_{\text{loc}}(\Omega \times \mathbb{R}^n)$ such that $A = \text{Op}(a') + R$, where R is an operator which has a kernel in $C_\infty(\Omega \times \Omega)$. The symbol a' is here defined modulo $S^{-\infty}_{\text{loc}}(\Omega \times \mathbb{R}^n)$. ∎

Definition 3.12 *In the situation of Proposition 3.11, we say that A is a pseudodifferential operator of order m on Ω, and the class of $S^m_{\text{loc}}(\Omega \times \mathbb{R}^n)/S^{-\infty}_{\text{loc}}(\Omega \times \mathbb{R}^n)$ is called the symbol of A.*

Proposition 3.13 *Let Ω, U be open subsets of $\mathbb{R}^n$ and P be a pseudodifferential operator of order m on U. If $\chi : \Omega \to U$ is a C_∞ change of coordinates, then the operator $P(\cdot \circ \chi^{-1}) \circ \chi$ is a pseudodifferential operator of order m on Ω.* ∎

3.1.3 Pseudodifferential operators on a manifold

Definition 3.14 *Let $m \in \mathbb{R}$ and V be a C_∞ manifold. A pseudodifferential operator of order m on V is a linear and continuous map $P : D(V) \to C_\infty(V)$ such that for every chart (U, φ) of V, the operator*

$$P_{(U,\varphi)} : D(\varphi(U)) \to C_\infty(\varphi(U)) \quad g \mapsto P(g \circ \varphi) \circ \varphi^{-1}$$

is a pseudodifferential operator of order m on $\varphi(U)$.

3.1.4 An application: pseudodifferential operators on Γ

By the following remark, we can restrict ourselves to the circle S.

Remark 3.15 *Let $m \in \mathbb{R}$. An operator*

$$P : C_\infty(\Gamma) \to C_\infty(\Gamma)$$

is a pseudodifferential operator of order m on Γ if $K_\gamma P K_\gamma^{-1}$ is a pseudodifferential operator of order m on S.

Notation 3.16 *For every real numbers α, β such that $0 < \beta - \alpha < 1$, we define the open subset $U_{\alpha,\beta}$ by*

$$U_{\alpha,\beta} := \{e^{2i\pi\theta} : \theta \in]\alpha, \beta[\}$$

and the map $\varphi_{\alpha,\beta}$ by

$$\varphi_{\alpha,\beta} : U_{\alpha,\beta} \to [0, 1[: \quad e^{2i\pi\theta} \mapsto \theta \bmod (1).$$

Using Definitions 3.14 and 3.12, a pseudodifferential operator of order m on $\mathcal{S}$ is a linear and continuous map $P : C_\infty(\mathcal{S}) \to C_\infty(\mathcal{S})$ such that for every chart $(U_{\alpha,\beta}, \varphi_{\alpha,\beta})$ of $\mathcal{S}$, the operator

$$P_{\alpha,\beta} : D(]\alpha, \beta[) \to C_\infty(]\alpha, \beta[) \quad g \mapsto P(g \circ \varphi_{\alpha,\beta}) \circ \varphi_{\alpha,\beta}^{-1}$$

is a pseudodifferential operator of order m on $]\alpha, \beta[$, i.e.

$$\varphi P_{\alpha,\beta} \psi \in \mathrm{Op}(S^m(\mathbb{R} \times \mathbb{R})), \quad \forall \varphi, \psi \in D(]\alpha, \beta[).$$

Definition 3.17 and the main of the following results follow [1] and [6].

Definition 3.17 *Let $\varepsilon \in]0, 1[$. We say that a function $a(x, \xi)$ which is 1-periodic on its first argument and belongs to $S^m(\mathbb{R} \times \mathbb{R})$ is a ε-symbol of a pseudodifferential operator P of order m on $\mathcal{S}$ if, $\forall u \in C_{\infty,1-\mathrm{per}}(\mathbb{R})$ for which there exists $\alpha, \beta \in \mathbb{R}$ such that*

$$\mathrm{supp}(u) \subset \cup_{\ell \in \mathbb{Z}}]\alpha + \ell, \beta + \ell[$$

with $\beta - \alpha = 1 - \varepsilon$, we have, $\forall \ell \in \mathbb{Z}$,

$$(Pu)(x) = (2\pi)^{-1} \int_{\mathbb{R}} d\xi \, e^{ix\xi} a(x, \xi) \int_{\alpha+\ell}^{\beta+\ell} dy \, e^{-iy\xi} u(y), \quad \forall x \in]\alpha + \ell, \beta + \ell[.$$

Proposition 3.18 *A pseudodifferential operator P on $\mathcal{S}$ is of order m if and only if, for all $s \in \mathbb{R}$, there is $C_s > 0$ such that*

$$||Pu||_{H^{s-m}_{1-\mathrm{per}}(\mathbb{R})} \leq C_s ||u||_{H^s_{1-\mathrm{per}}(\mathbb{R})}, \quad \forall u \in H^s_{1-\mathrm{per}}(\mathbb{R}).$$

∎

3.1.4.1 About the kernel

Proposition 3.19 *If P is a pseudodifferential operator of order $m < -1$ on $\mathcal{S}$, then P is an integral operator with a continuous and 1-periodic kernel. To be more precise, we have*

$$(Pu)(x) = \int_{\delta}^{\delta+1} K(x, y) u(y) dy, \quad \forall u \in C_{\infty,1-\mathrm{per}}(\mathbb{R})$$

with δ chosen arbitrary in $\mathbb{R}$, K given by

$$K(x, y) = \sum_{p \in \mathbb{Z}} e^{2i\pi px} \sum_{n \in \mathbb{Z}} a_{p,n} e^{-2i\pi ny},$$

and where the coefficients $a_{p,n}$ are the Fourier coefficients of $P(e^{2i\pi n})$

$$v_n = \int_0^1 v(y)e^{-2i\pi ny}dy, \quad \forall v \in L^2(]0,1[), \forall n \in \mathbb{Z};$$
$$a_{p,n} = \left(P(e^{2i\pi n})\right)_p, \quad \forall n, p \in \mathbb{Z}.$$

PROOF. a) We start with the proof of the continuity of K. Let $M := -m = |m| > 1$.

1) For every $n \in \mathbb{Z}$, the distribution $\sum_{k\in\mathbb{Z}} \delta_{n,k} e^{2i\pi k}$ belongs to $H^{-M}_{1-\text{per}}(\mathbb{R})$. It follows that we have

$$\sum_{n\in\mathbb{Z}} (1+n^2)^{\frac{M}{2}} \sum_{p\in\mathbb{Z}} |a_{p,n}|^2 \le C^2_{-M} \sum_{n\in\mathbb{Z}} (1+n^2)^{-\frac{M}{2}} := C^{(1)}$$

and there is a constant $C^{(2)}$ such that

$$\sum_{n\in\mathbb{Z}} |a_{p,n}| \le C^{(2)}, \quad \forall p \in \mathbb{Z}.$$

For every $p \in \mathbb{Z}$, we define the function h_p by $h_p(y) := \sum_{n\in\mathbb{Z}} a_{p,n} e^{-2i\pi ny}$. Those functions are clearly continuous on $\mathbb{R}$.

2) For every $y \in \mathbb{R}$, the distribution $\sum_{k\in\mathbb{Z}} e^{2i\pi k(\cdot - y)}$ belongs to $H^{-\frac{M}{2}}_{1-\text{per}}(\mathbb{R})$. It follows that we have

$$\sum_{p\in\mathbb{Z}} (1+p^2)^{\frac{M}{2}} |h_p(y)|^2 \le C^2_{-\frac{M}{2}} \sum_{k\in\mathbb{Z}} (1+k^2)^{-\frac{M}{2}} := C^{(3)},$$

where the constant $C^{(3)}$ does not depend on y. So, we get $\sum_{p\in\mathbb{Z}} |h_p(y)| \le C^{(4)}$, with $C^{(4)}$ independent of y. It follows that, with y fixed, the series

$$\sum_{p\in\mathbb{Z}} e^{2i\pi p} \sum_{n\in\mathbb{Z}} a_{p,n} e^{-2i\pi ny} = \sum_{p\in\mathbb{Z}} e^{2i\pi p} h_p(y)$$

defines a continuous function on $\mathbb{R}$.

3) For any y, y' fixed in $\mathbb{R}$, the distribution $\sum_{k\in\mathbb{Z}} (e^{-2i\pi ky} - e^{-2i\pi ky'}) e^{2i\pi k}$ also belongs to $H^{-\frac{M}{2}}_{1-\text{per}}(\mathbb{R})$. Therefore, if we define $\varepsilon > 0$ by $\varepsilon := \inf\{1, \frac{M-1}{4}\}$, we have

$$\begin{aligned}\sum_{p\in\mathbb{Z}} (1+p^2)^{\frac{M}{2}} |h_p(y) - h_p(y')|^2 &\le C^2_{-\frac{M}{2}} \sum_{k\in\mathbb{Z}} (1+k^2)^{-\frac{M}{2}} |e^{-2i\pi ky} - e^{-2i\pi ky'}|^2 \\ &\le C^{(5)} |y - y'|^{2\varepsilon}\end{aligned}$$

where the constant $C^{(5)}$ does not depend on y and y'.

4) For every x, y, $y' \in \mathbb{R}$, we have

$$|K(x,y) - K(x,y')| \le C^{(6)} |y - y'|^{\varepsilon}.$$

So, we obtain the continuity of K on $\mathbb{R}^2$.

b) Now let us prove that P have K as kernel. This is direct because, $\forall p \in \mathbb{Z}$, we have

$$(Pu)_p = \sum_{k\in\mathbb{Z}} a_{p,k} u_k = \int_\delta^{1+\delta} h_p(y) u(y) dy, \quad \forall \delta \in \mathbb{R}.$$

∎

Proposition 3.20 *The proposition 3.19 can be extended to functions $u \in L^2(]0,1[)$.* ∎

In the two next propositions, we use the same notations as in Proposition 3.19.

Proposition 3.21 *If P is a pseudodifferential operator of order $m < -2$, then the summation order which appears in the expression of K can be arbitrary chosen.*

PROOF. For every $n \in \mathbb{Z}$, the distribution $\sum_{k\in\mathbb{Z}} \delta_{n,k} e^{2i\pi k}$ belongs to $H_{1-\mathrm{per}}^{-\frac{M}{2}}(\mathbb{R})$. Therefore, we have

$$\sum_{p\in\mathbb{Z}} (1+p^2)^{\frac{M}{2}} |a_{p,n}|^2 \le C_{-\frac{M}{2}}^2 (1+n^2)^{-\frac{M}{2}},$$

which implies

$$\sum_{n\in\mathbb{Z}} \sum_{p\in\mathbb{Z}} |a_{p,n}| \le C^{(7)} \sum_{n\in\mathbb{Z}} (1+n^2)^{-\frac{M}{4}}.$$

∎

Proposition 3.22 *If P is a pseudodifferential operator on S of order $m < -1$, then the following assertions are equivalent*

a) $m = -\infty$;

b) the sequence $(a_{p,n})_{p,n\in\mathbb{Z}}$ is fast decaying, i.e. $\forall k \in \mathbb{Z}$, there exists a constant $C_k^{'} > 0$ such that

$$|a_{p,n}| \le C_k^{'} (1+p^2)^{-\frac{k}{2}} (1+n^2)^{-\frac{k}{2}} \quad \forall p, n \in \mathbb{Z};$$

c) $K(x,y) \in C_\infty(\mathbb{R} \times \mathbb{R})$.

PROOF.

Let us prove that a) implies b). For every $n \in \mathbb{Z}$, the distribution $\sum_{k\in\mathbb{Z}} \delta_{n,k} e^{2i\pi k}$ belongs to $\cap_{s\in\mathbb{R}} H_{1-\mathrm{per}}^s(\mathbb{R})$. Hence we have

$$\sum_{p\in\mathbb{Z}} (1+p^2)^{s+M} |a_{p,n}|^2 \le C_s^2 (1+n^2)^s, \quad \forall s \in \mathbb{R},\ \forall M \in \mathbb{R}.$$

The conclusion immediately follows.

Let us prove that b) implies a). Let $m, s \in \mathbb{R}$. It is easy to show that there exists a constant $C^{(8)}$ such that

$$\sum_{p\in\mathbb{Z}} (1+p^2)^{s-m} \Big| \sum_{k\in\mathbb{Z}} a_{p,k} u_k \Big|^2 \le C^{(8)} \sum_{k\in\mathbb{Z}} (1+k^2)^s |u_k|^2, \quad \forall u \in H_{1-\mathrm{per}}^s(\mathbb{R}).$$

Finally, the points b) and c) are equivalent because the series

$$\sum_{p\in\mathbb{Z}} p^\alpha e^{2i\pi p x} \sum_{n\in\mathbb{Z}} n^\beta a_{p,n} e^{-2i\pi n y}$$

converges uniformly on every compact set of $\mathbb{R}^2$, $\forall \alpha \in \mathbb{N}$, $\forall \beta \in \mathbb{N}$ if and only if the sequence $(a_{p,n})_{p,n\in\mathbb{Z}}$ is fast decaying. ∎

3.1.4.2 Relation between operators and symbols

We begin this subsection with a lemma that gives the structure of any ε-symbol of a pseudodifferential operator. The main important properties connecting operators and symbols will be deduced from it.

Lemma 3.23 *Let P a pseudodifferential operator of order $m < -1$ on $\mathcal{S}$, K its kernel (given by Proposition 3.19) and $\varepsilon \in]0,1[$. The following assertions are equivalent*

1. $a \in S^m(\mathbb{R}\times\mathbb{R})$, *is 1-periodic on its first argument and is an ε-symbol of P;*
2. $a(x,\xi) = \int_{\mathbb{R}} e^{-i\xi t} L(x,t)dt$ *where L is a function on $\mathbb{R}\times\mathbb{R}$ which satisfies*
   (a) $L(x,t) = K(x,x-t)$, $\forall x,t \in \mathbb{R}$ *such that* $|t| \leq 1-\varepsilon$;
   (b) $L \in C_\infty(\mathbb{R}\times\mathbb{R}_0)$;
   (c) $\forall p,q \in \mathbb{N}$, $\mathrm{D}_x^p \mathrm{D}_t^q L(x,t)$ *is a 1-periodic function on its first argument such that, $\forall N \in \mathbb{N}$, there is $C_{p,q,N>0}$ such that*

$$|\mathrm{D}_x^p \mathrm{D}_t^q L(x,t)| \leq C_{p,q,N}|t|^{-N}$$

when $|t| \geq 1$.

PROOF. Let us prove that condition 1 implies condition 2. Since $m < -1$, we can define L by

$$L(x,t) := (2\pi)^{-1}\int_{\mathbb{R}} e^{it\xi}a(x,\xi)d\xi$$

If $t \in \mathbb{R}_0$, $x \in \mathbb{R}$, we have

$$L(x,t) = (2\pi)^{-1}(-it)^{-k}\int_{\mathbb{R}} e^{it\xi}\mathrm{D}_\xi^k a(x,\xi)d\xi, \quad \forall k \in \mathbb{N},$$

hence we obtain all the required conditions on L.

Let us prove that condition 2 implies condition 1. Let α,β be real numbers such that $\beta-\alpha = 1-\varepsilon$ and u be a function of $C_{\infty,1\text{-per}}(\mathbb{R})$ which satisfies $\mathrm{supp}(u) \subset \cup_{\ell\in\mathbb{Z}}]\alpha+\ell,\beta+\ell[$. For any $x \in]\alpha,\beta[$, we have

$$\begin{aligned}(Pu)(x) &= \int_\alpha^\beta K(x,y)u(y)dy\\ &= \int_\alpha^\beta L(x,x-y)u(y)dy\\ &= (2\pi)^{-1}\int_{\mathbb{R}} e^{i\xi x}a(x,\xi)\widehat{u\chi_{[\alpha,\beta]}}(\xi)\,d\xi.\end{aligned}$$

So, the main point is now to prove that the function

$$a(x,\xi) = \int_{\mathbb{R}} e^{-i\xi t}L(x,t)dt$$

3.1.4.3 Main result

Proposition 3.29 *Let P be a pseudodifferential operator of order m on S which admits a as ε-symbol (ε fixed in $]0,1[$). Then, there exists an operator T of order $-\infty$ such that*

$$((P-T)u)(x) = \sum_{n\in\mathbb{Z}} e^{2i\pi nx} a(x, 2\pi n) u_n, \quad \forall u \in C_{\infty,1-\text{per}}(\mathbb{R})$$

PROOF. We begin with the case $m < -1$. Let K be the kernel of P. Let δ_1, δ_2 be real numbers such that $0 < \delta_1 < \delta_2 < \frac{1}{2}$ and $\rho(t)$ be a function belonging to $D(\mathbb{R})$ which is equal to 1 if $|t| \le \delta_1$ and to 0 if $|t| \ge \delta_2$. We can define the operator P_1 by

$$(P_1 u)(x) := \int_{\mathbb{R}} K(x,y)\rho(x-y)u(y)dy, \quad u \in C_{\infty,1-\text{per}}(\mathbb{R})$$

and the operator T_1 by $T_1 := P - P_1$. The operator T_1 is of order $-\infty$. Indeed, using Proposition 3.19, we have

$$(T_1 u)(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} K'(x,y)u(y)dy$$

with

$$K'(x,y) := K(x,y)(1-\rho(x-y)) \in C_\infty(\mathbb{R}\times\mathbb{R}).$$

Using Lemma 3.23, the function

$$a^{(1)}(x,\xi) := \int_{\mathbb{R}} e^{i(y-x)\xi} K(x,y)\rho(x-y)dy$$

is a $(1-\delta_1)$-symbol of P. For every $n \in \mathbb{Z}$, we have $a^{(1)}(x, 2\pi n) = e^{-2i\pi nx}(P_1(e^{2i\pi n \cdot}))(x)$. The continuity of the operator P_1 gives

$$(P_1 u)(x) = \sum_{n\in\mathbb{Z}} e^{2i\pi nx} a^{(1)}(x, 2\pi n) u_n, \quad \forall u \in C_{\infty,1-\text{per}}(\mathbb{R}).$$

The symbol $a(x,\xi) - a^{(1)}(x,\xi)$ belongs to $S^{-\infty}(\mathbb{R}\times\mathbb{R})$, so the $\mathcal{K}$ function defined by

$$\mathcal{K}(x,y) := \sum_{n\in\mathbb{Z}} e^{2i\pi n(x-y)} \left(a(x,2\pi n) - a^{(1)}(x, 2\pi n)\right)$$

belongs to $C_\infty(\mathbb{R}\times\mathbb{R})$ and allows us to define a pseudodifferential operator T_2 of order $-\infty$ on S

$$(T_2 u)(x) = \int_0^1 dy\, \mathcal{K}(x,y)u(y), \quad u \in C_{\infty,1-\text{per}}(\mathbb{R}).$$

For any $u \in C_{\infty,1-\text{per}}(\mathbb{R})$, we have

$$(T_2 u)(x) = \sum_{n\in\mathbb{Z}} e^{2i\pi nx} \left(a(x,2\pi n) - a^{(1)}(x,2\pi n)\right) u_n.$$

So, we conclude by setting $T := T_1 - T_2$

$$((P-T)u)(x) = ((P_1+T_2)u)(x) = \sum_{n\in\mathbb{Z}} e^{2i\pi nx} a(x, 2\pi n) u_n.$$

Let us now consider the case $m \geq -1$. From m, we define $\ell \in \mathbb{N}_0$ and $s \in \mathbb{R}$ as in Proposition 3.28. One gets $P = P'\left(1 - (\frac{1}{2\pi}\mathrm{D})^2\right)^{\ell}$ where P' is a pseudodifferential operator of order s on S which admits the ε-symbol $a'(x,\xi) := \frac{a(x,\xi)}{\left(1+(\frac{\xi}{2\pi})^2\right)^{\ell}}$. The result immediately follows. ∎

3.2 Céa lemma applied to a large class of pseudodifferential operators

Our main purpose is to show how the Θ_m-spline wavelets can be used to obtain efficient and very stable methods of resolution for some kind of boundary equations. To this end, we present a result providing an estimate of Céa type for strongly elliptic pseudodifferential operators with constant coefficients. It concerns splines of any order (see also [39]). The technique used here presents a new expression of the condition leading to the estimate of Céa type. This expression gives then an easy description of the relations between the degree of the splines and the meshes.

3.2.1 The Dirichlet problem for the Laplace's equation

From [26], Sobolev spaces are the natural spaces to set the Dirichlet problem.

Proposition 3.30 *If $s > \frac{1}{2}$ and $f \in H^{s-\frac{1}{2}}(\Gamma)$, then the problem*

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega \\ u_{|\Gamma} &= f \end{cases}$$

has a unique solution $u \in H^s(\Omega)$. ∎

In practice, the potential's methods are widely used to solve this problem.

Definition 3.31 *Let $\beta \in L^2(\Gamma)$ and $E(x) := -\frac{1}{2\pi}\ln(|x|)$ (fundamental solution for $-\Delta$). The single layer potential of density β on Γ is defined by the integral*

$$(V\beta)(x) := \int_\Gamma E(x-y)\,\beta(y)\,d\sigma(y) = -\frac{1}{2\pi}\int_\Gamma \ln(|x-y|)\,\beta(y)\,d\sigma(y), \quad x \in \mathbb{R}^2,$$

and the double layer potential of density β on Γ by

$$(K\beta)(x) := \int_\Gamma \frac{\partial E(x-y)}{\partial \nu_y}\,\beta(y)\,d\sigma(y) = \frac{1}{2\pi}\int_\Gamma \frac{(x-y).\nu_y}{|x-y|^2}\,\beta(y)\,d\sigma(y), \quad x \in \mathbb{R}^2,$$

where ν is a unit normal to the boundary.

Proposition 3.32 *For every $\beta \in L^2(\Gamma)$, the functions $V\beta$ and $K\beta$ are harmonic in $\mathbb{R}^2 \setminus \Gamma$.* ∎

Remark 3.33 *Let us mention the physical significance of those potentials. The single layer potential is the potential of charges distributed over Γ with density β and the double layer potential is the potential of dipoles distributed over Γ with density β and oriented in the direction of the chosen normal (see [26]). We clearly don't want to develop here this physical aspect.*

Remark 3.34 *In the following, we will systematically choose ν to be the unit inward normal to the boundary.*

If we use the single layer potential representation of u

$$u(x) = -\frac{1}{2\pi} \int_\Gamma \ln(|x-y|)\, v(y)\, d\sigma(y), \quad x \in \Omega,$$

to solve the Dirichlet problem, the boundary equation for v is simply $Vv = f$ with

$$(Vv)(x) = -\frac{1}{2\pi} \int_\Gamma \ln(|x-y|)\, v(y)\, d\sigma(y), \quad x \in \Gamma.$$

If we search the solution as a double layer potential

$$u(x) = \frac{1}{2\pi} \int_\Gamma \frac{(x-y).\nu_y}{|x-y|^2}\, w(y)\, d\sigma(y), \quad x \in \Omega,$$

the boundary equation for w is $(\frac{1}{2} + K)w = f$ with

$$Kw(x) = \frac{1}{2\pi} \int_\Gamma \frac{(x-y).\nu_y}{|x-y|^2}\, w(y)\, d\sigma(y), \quad x \in \Gamma.$$

Let us mention a classical important result (see [29]).

Proposition 3.35 *The operator $V : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is an isomorphism for every s if and only if the analytic capacity of Ω is not 1. Moreover $\frac{1}{2} + K : H^s(\Gamma) \to H^s(\Gamma)$ is an isomorphism for every $s \in \mathbb{R}$. Since the boundary is smooth, K is a compact operator from $H^s(\Gamma)$ to $H^s(\Gamma)$.* ∎

3.2.2 A more general setting

These two boundary operators V and $\frac{1}{2} + K$ can be considered on functions $u \in L^2_{1-\text{per}}(\mathbb{R})$. As it is proved in the two following points, they are particular cases of the following classical pseudodifferential operators with constant coefficients in the periodic setting

$$P = b_+ Q_+^\beta + b_- Q_-^\beta + K_0$$

where $b_+, b_- \in \mathbb{C}$, $\beta \in \mathbb{R}$,

$$Q_+^\beta u(x) = \sum_{k\in\mathbb{Z}_0} |k|^\beta u_k e^{2ik\pi x}, \quad Q_-^\beta u(x) = \sum_{k\in\mathbb{Z}_0} \operatorname{sgn}(k)|k|^\beta u_k e^{2ik\pi x}$$

and, for all $r \in \mathbb{R}$, K_0 is a compact operator from $H^{r+\beta}_{1-\mathrm{per}}(\mathbb{R})$ into $H^r_{1-\mathrm{per}}(\mathbb{R})$.

By Propositions 3.29, 3.25 and 3.26, the operator $b_+Q_+^\beta + b_-Q_-^\beta$ is a pseudodifferential operator of order β on $\mathcal{S}$ of ε-symbol

$$a(x,\xi) = (b_+ + b_-\operatorname{sgn}(\xi))\left|\frac{\xi}{2\pi}\right|^\beta.$$

As announced above, let us consider the particular cases of the double and the single layer potentials.

a) For the double layer potential, we have

$$\frac{1}{2} + K : \; u \in L^2_{1-\mathrm{per}}(\mathbb{R}) \mapsto \frac{1}{2}\sum_{k\in\mathbb{Z}_0} u_k e^{2i\pi k} + \frac{u_0}{2} + Ku,$$

which corresponds to $\beta = 0$, $b_- = 0$, $b_+ = \frac{1}{2}$ and

$$K_0 : \; u \mapsto \frac{u_0}{2} + Ku$$

since K is compact from $H^r(\Gamma)$ into itself, $\forall r \in \mathbb{R}$.

b) For the single layer potential, if γ^0 is a parametrization of Γ proportional to arc length and defined in $[0,1]$, we have

$$\begin{aligned} V : \; u \in L^2_{1-\mathrm{per}}(\mathbb{R}) &\mapsto -\frac{1}{2\pi}\int_0^1 \ln(|\gamma^0(\cdot) - \gamma^0(s)|)u(s)ds \\ &= -\frac{1}{2\pi}\int_0^1 \ln\Big(\frac{|\gamma^0(\cdot) - \gamma^0(s)|}{|e^{2i\pi} - e^{2i\pi s}|}\Big)u(s)ds - \frac{1}{2\pi}\int_0^1 \ln(|e^{2i\pi} - e^{2i\pi s}|)u(s)ds \\ &= K_1 u + V_1 u \end{aligned}$$

where K_1 is a compact operator from $H^r_{1-\mathrm{per}}(\mathbb{R})$ into $C_{\infty\,1-\mathrm{per}}(\mathbb{R})$, $\forall r \in \mathbb{R}$ (K_1 is an integral operator with a C_∞ kernel). Since the operator

$$V_1 : u \mapsto -\frac{1}{2\pi}\int_0^1 \ln(|e^{2i\pi} - e^{2i\pi s}|)u(s)\,ds$$

is such that

$$\begin{aligned} V_1(e^{2i\pi k}) &= -\frac{1}{2\pi}\int_0^1 \ln(|e^{2i\pi} - e^{2i\pi s}|)e^{2i\pi k s}\,ds \\ &= -\frac{e^{2i\pi k}}{2\pi}\int_0^1 \ln(|1 - e^{2i\pi s}|)e^{2i\pi k s}\,ds \\ &= \frac{e^{2i\pi k}}{2\pi}\int_0^1 e^{2i\pi k s}\sum_{\ell=1}^{+\infty}\frac{\cos(2\pi\ell s)}{\ell}\,ds \\ &= \begin{cases} \dfrac{e^{2i\pi k}}{4\pi|k|} & \text{if } k \in \mathbb{Z}_0 \\ 0 & \text{if } k = 0, \end{cases} \end{aligned}$$

we get

$$Vu = K_1 u + \frac{1}{4\pi} \sum_{k \in \mathbb{Z}_0} \frac{1}{|k|} u_k e^{2i\pi k}.$$

This corresponds to $\beta = -1$, $b_- = 0$, $b_+ = \frac{1}{4\pi}$ and $K_0 = K_1$.

3.2.3 Céa lemma

Let us present an adapted version of the Céa lemma (see [33]).

Lemma 3.36 *(Céa) Let X, Y be Banach spaces and P a bijective operator of $L(X, Y)$. Let $(V_j)_{j \in \mathbb{N}_0}$ (resp. $(T_j)_{j \in \mathbb{N}_0}$) a sequence of subsets of X (resp. Y') such that $dim(V_j) = dim(T_j) < +\infty$ for every $j \in \mathbb{N}_0$. Assume that the following two conditions are satisfied*

1. *for every $j \in \mathbb{N}_0$, there is an operator $P_j \in L(Y', T_j)$ such that $P_j(f) \to f$ in Y' for every $f \in Y'$;*
2. *there are $c > 0$ and a compact operator $K \in L(X, X)$ such that*

$$\sup_{v \in T_j, \|v\|_{Y'}=1} |v(Pu)| \geq c\|u\|_X - \|Ku\|_X$$

for every $u \in V_j$.

Then there is $J > 0$ such that, for every $j \geq J$ and $u \in X$, the system

$$v(Pu_j) = v(Pu), \quad v \in T_j$$

has a unique solution $u_j \in V_j$. Moreover, there is $C > 0$ such that

$$\|u - u_j\|_X \leq C \inf_{w \in V_j} \|u - w\|_X.$$

∎

In this lemma, the second condition is called the *coercivity condition*.

In the next proposition, we keep all the notations of Lemma 3.36 and we follow the proof of [5].

Proposition 3.37 *If X, Y are Hilbert spaces and if the conditions 1. and 2. of Lemma 3.36 are satisfied, then there is $J_1 \in \mathbb{N}_0$ such that for every $j \geq J_1$, the estimation*

$$\sup_{v \in T_j, \|v\|_{Y'}=1} |v(Pu)| \geq c\|u\|_X - \|Ku\|_X, \quad \forall u \in V_j$$

remains valid with $K = 0$ and a smaller constant c.

PROOF. Let $j \in \mathbb{N}_0$ and $u \in V_j$ such that $Ku \neq 0$. There is $v_1 \in T_j$ such that $\|v_1\|_{Y'} = 1$ and $|v_1(Pu)| \geq c_1\|u\|_X - \|Ku\|_X$ with $0 < c_1 < c$. Let $\theta \in [0, 2\pi[$ such that $|v_1(Pu)| = v_1(Pu)e^{-i\theta}$. The function $v_2 := e^{-i\theta}v_1 \in T_j$ satisfies $\|v_2\|_{Y'} = 1$ and $v_2(Pu) \geq c_1\|u\|_X - \|Ku\|_X$. We define the operator T by

$$T : Y \to Y' \quad y \mapsto \langle \cdot , y\rangle_Y.$$

It is an isometric bijection between Y and Y'.

For every $\ell \in \mathbb{N}_0$, we define the operator S_ℓ by

$$S_\ell := P^*T^{-1}(I - P_\ell)T(P^*)^{-1}K^*.$$

We get $\lim_{\ell\to+\infty} \|S_\ell\| = 0$. Indeed, the sequence $I - P_\ell$ converges strongly to 0 and K^* is a compact operator (see [5]).

We define V by

$$V := v_2 + P_jT(P^*)^{-1}K^*\frac{Ku}{\|Ku\|_X}.$$

One gets $V \in T_j$, $\|V\|_{Y'} \leq 1 + \|P^{-1}\|\,\|K\|$ and

$$\begin{aligned}
&|V(Pu)| \\
&\geq \mathcal{R}e(V(Pu)) \\
&= v_2(Pu) + \mathcal{R}e\Big(\Big(T(P^*)^{-1}K^*\frac{Ku}{\|Ku\|_X}\Big)(Pu)\Big) \\
&\quad -\mathcal{R}e\Big(\Big((I - P_j)T(P^*)^{-1}K^*\frac{Ku}{\|Ku\|_X}\Big)(Pu)\Big) \\
&\geq v_2(Pu) + \|Ku\|_X - \Big|\Big((I - P_j)T(P^*)^{-1}K^*\frac{Ku}{\|Ku\|_X}\Big)(Pu)\Big| \\
&= v_2(Pu) + \|Ku\|_X - \Big|\langle Pu, T^{-1}(I - P_j)T(P^*)^{-1}K^*\frac{Ku}{\|Ku\|_X}\rangle_Y\Big| \\
&\geq (c_1 - \|S_j\|)\|u\|_X.
\end{aligned}$$

So we can conclude since we have

$$\sup_{v\in T_j, \|v\|_{Y'}=1} |v(Pu)| \geq \frac{1}{\|V\|_{Y'}}|V(Pu)| \geq \frac{c_1}{2(1 + \|P^{-1}\|\,\|K\|)}\|u\|_X$$

$j \geq J_1$, $J_1 \in \mathbb{N}_0$ chosen such that $\|S_\ell\| \leq \frac{c_1}{2}$ if $\ell \geq J_1$. ∎

2.4 Application of the Céa lemma

apply the Céa lemma to our operator P, we naturally choose $X = H^{s+\beta}_{1-\text{per}}(\mathbb{R})$ and $= H^s_{1-\text{per}}(\mathbb{R})$ for s fixed in $\mathbb{R}$ and we consider the operator

$$P = b_+Q_+^\beta + b_-Q_-^\beta + K_0$$

X into Y.

For the spaces V_j $(j \in \mathbb{N}_0)$, we choose the spaces $\mathcal{V}_{j,\delta}^{(r+1)}$, for r fixed in $\mathbb{N}$ and δ fixed in $[0,1[$. The condition $\mathcal{V}_{j,\delta}^{(r+1)} \subset H_{1-\text{per}}^{s+\beta}(\mathbb{R})$ requires $s+\beta < r+\frac{1}{2}$.

For every $j \in \mathbb{N}_0$, we define the space T_j by

$$T_j := \{\langle \cdot \,, g\rangle_{H_{1-\text{per}}^m(\mathbb{R})} : g \in \mathcal{V}_j^{(2m)}\}$$

with m fixed in $\mathbb{N}_0$. The condition $\mathcal{V}_j^{(2m)} \subset H_{1-\text{per}}^{2m-s}(\mathbb{R})$ is satisfied if $s > \frac{1}{2}$.

For every $j \in \mathbb{N}_0$, we define P_j as the orthogonal projector from Y' onto T_j.

If $s > 0$, the first condition of Lemma 3.36 is a consequence of the Propositions 3.38 and 3.39 below.

Proposition 3.38 *If $s > \frac{1}{2}$, the union $\cup_{j \in \mathbb{N}_0} T_j$ is dense in Y'.*

PROOF. Let $e' \in Y'$. By Riesz lemma, there is $F \in Y$ such that $e'(\cdot) = \langle \cdot \,, F\rangle_Y$. For $g \in \cup_{j\in\mathbb{N}_0}\mathcal{V}_j^{(2m)}$, we have

$$\begin{aligned}
\|e' - \langle \cdot \,, g\rangle_{H_{1-\text{per}}^m(\mathbb{R})}\|_{Y'} &= \sup_{\|f\|_Y \le 1} |e'(f) - \langle f \,, g\rangle_{H_{1-\text{per}}^m(\mathbb{R})}| \\
&= \sup_{\|f\|_Y \le 1} |\langle f \,, F\rangle_Y - \langle f \,, g\rangle_{H_{1-\text{per}}^m(\mathbb{R})}| \\
&= \sup_{\|f\|_Y \le 1} |\langle f \,, F\rangle_Y - \langle f \,, G\rangle_Y| \\
&= \|F - G\|_Y \\
&= \sqrt{F_0 - g_0 + \sum_{k\in\mathbb{Z}_0} |k|^{2s}\,\left|F_k - |k|^{2(m-s)} g_k\right|^2} \\
&= \sqrt{F_0 - g_0 + \sum_{k\in\mathbb{Z}_0} |k|^{2(2m-s)}\,\left||k|^{2(s-m)} F_k - g_k\right|^2} \\
&= \|F' - g\|_{H_{1-\text{per}}^{2m-s}(\mathbb{R})}
\end{aligned}$$

with $G := g_0 + \sum_{k\in\mathbb{Z}_0} |k|^{2(m-s)} g_k e^{-2i\pi k\cdot} \in Y$ and $F' := F_0 + \sum_{k\in\mathbb{Z}_0} |k|^{2(s-m)} F_k e^{-2i\pi k\cdot} \in H_{1-\text{per}}^{2m-s}(\mathbb{R})$. We conclude using Proposition 2.16. ∎

Proposition 3.39 *For every $f \in Y'$, we have $P_j(f) \to_{Y'} f$ if $j \to +\infty$.*

PROOF. Let $f \in Y'$ and $\varepsilon > 0$. By Proposition 3.38, there exists $j_0 \in \mathbb{N}_0$ and $f_0 \in T_{j_0}$ such that $\|f - f_0\|_{Y'} \le \frac{\varepsilon}{2}$. For every $j \ge j_0$, it is clear that $P_j(f_0) = f_0$. Then we have

$$\|f - P_j(f)\|_{Y'} \le \|f - f_0\|_{Y'} + \|P_j(f_0) - P_j(f)\|_{Y'} \le \varepsilon.$$

∎

So, if the coercivity condition is satisfied, then there is $J > 0$ such that, for every $j \ge J$ and $u \in H_{1-\text{per}}^{s+\beta}(\mathbb{R})$, the Galerkin system

$$\langle Pu_j \,, g\rangle_{H_{1-\text{per}}^m(\mathbb{R})} = \langle Pu \,, g\rangle_{H_{1-\text{per}}^m(\mathbb{R})}, \quad g \in \mathcal{V}_j^{(2m)}$$

has a unique solution $u_j \in \mathcal{V}_{j,\delta}^{(r+1)}$. Moreover, there is $C > 0$ such that

$$\|u - u_j\|_{H^{s+\beta}_{1-\text{per}}(\mathbb{R})} \leq C \inf_{w \in \mathcal{V}_{j,\delta}^{(r+1)}} \|u - w\|_{H^{s+\beta}_{1-\text{per}}(\mathbb{R})},$$

Therefore, the main point is to check the coercivity condition. It is the aim of the next subsection.

3.2.5 Coercivity

In a first step, we give in Theorem 3.41 a necessary and sufficient condition of coercivity for the operator

$$P : H^{s+\beta}_{1-\text{per}}(\mathbb{R}) \to H^{s}_{1-\text{per}}(\mathbb{R}) : \ u \mapsto b_+ Q_+^\beta u + b_- Q_-^\beta u + u_0.$$

The following lemma will be useful to prove this theorem.

Lemma 3.40 *Let $\beta \in \mathbb{R}$, $b_+, b_- \in \mathbb{C}$, $r \in \mathbb{N}$ such that $r > \beta$ and $\theta \in [0, 2\pi[$. We consider the function $a \in]0,1[\mapsto \mathcal{E}(a,\theta,r)$ where*

$$\begin{aligned}\mathcal{E}(a,\theta,r) &:= \sum_{p=-\infty}^{+\infty} e^{-ip\theta}(b_+ + b_-\text{sgn}(p+a))\frac{|p+a|^\beta}{(p+a)^{r+1}} \\ &= c_+\mathcal{E}_+(a,\theta,r) + (-1)^{r+1}c_-\mathcal{E}_-(a,\theta,r)\end{aligned}$$

with

$$\mathcal{E}_+(a,\theta,r) = \sum_{p=0}^{+\infty} \frac{e^{-ip\theta}}{(p+a)^{r+1-\beta}}, \quad \mathcal{E}_-(a,\theta,r) = \sum_{p=-\infty}^{-1} \frac{e^{-ip\theta}}{(-p-a)^{r+1-\beta}}$$

and $c_+ := b_+ + b_-$, $c_- := b_+ - b_-$. We define the real number γ by

$$\gamma := \inf\{\mathcal{R}e(c_+), \mathcal{R}e(c_-)\}.$$

1) This function $\mathcal{E}$ is continuous and satisfies

$$\mathcal{E}(a,\theta,r) = \frac{1}{2\Gamma(r+1-\beta)} \int_0^{+\infty} \frac{t^{r-\beta}N^{(r)}(t,a,\theta)}{\cosh(t) - \cos(\theta)}\,dt$$

with

$$N^{(r)}(t,a,\theta) := c_+e^{-at}(e^t - e^{i\theta}) + (-1)^{r+1}c_-e^{at}(e^{i\theta} - e^{-t}).$$

We also have

$$\begin{aligned}\lim_{a\to 0^+} a^{r+1-\beta}\mathcal{E}_+(a,\theta,r) &= 1, \\ \lim_{a\to 1^-} (1-a)^{r+1-\beta}\mathcal{E}_-(a,\theta,r) &= e^{i\theta}.\end{aligned}$$

2) Assume ($b_- = 0$ and $b_+ \neq 0$) or ($b_+, b_- \in \mathbb{R}$ and $\gamma > 0$). If r is odd (respectively r is even), then

$$\mathcal{E}(a, \theta, r) \neq 0 \ \forall a \in]0,1[\Leftrightarrow \theta \neq \pi \text{ (respectively } \theta \neq 0).$$

3) Assume $\gamma > 0$. If r is odd (respectively r is even), then

$$\theta = 0 \text{ (respectively } \theta = \pi) \Rightarrow \mathcal{E}(a, \theta, r) \neq 0 \ \forall a \in]0,1[.$$

PROOF. See [8]. ∎

Theorem 3.41 *Let m be a strictly positive integer, $r \in \mathbb{N}$, $s, \beta \in \mathbb{R}$, $b_+, b_- \in \mathbb{C}$, $\delta \in [0,1[$, P the pseudodifferential operator*

$$P : H^{s+\beta}_{1-\text{per}}(\mathbb{R}) \to H^{s}_{1-\text{per}}(\mathbb{R}) : u \mapsto b_+ Q^\beta_+ u + b_- Q^\beta_- u + u_0$$

and assume that the conditions $s > \frac{1}{2}$ and $r + \frac{1}{2} > s + \beta$ are satisfied. Then there is $c > 0$ such that

$$\sup_{g \in \mathcal{V}_j^{(2m)},\, \|g\|_{H^{2m-s}_{1-\text{per}}(\mathbb{R})} = 1} |\langle Pf, g\rangle_{H^m_{1-\text{per}}(\mathbb{R})}| \geq c \|f\|_{H^{s+\beta}_{1-\text{per}}(\mathbb{R})}$$

for every $j \in \mathbb{N}$ and every $f \in \mathcal{V}_{j,\delta}^{(r+1)}$ if and only if $b_+ \neq \pm b_-$ and the function

$$a \mapsto \sigma_P^{(r)}(a, \theta) := \int_0^{+\infty} \frac{t^{r-\beta} N^{(r)}(t, a, \theta)}{\cosh(t) - \cos(\theta)}\, dt$$

with

$$N^{(r)}(t, a, \theta) := (b_+ + b_-) e^{-at}(e^t - e^{i\theta}) + (-1)^{r+1}(b_+ - b_-) e^{at}(e^{i\theta} - e^{-t}).$$

and $\theta := 2\pi\delta$ does not vanish in $]0,1[$.

PROOF. See [8]. ∎

In order to use collocation methods, we also need a similar result for the norm $\|\cdot\|_s$.

Proposition 3.42 *Let m be a strictly positive integer, $r \in \mathbb{N}$, $s, \beta \in \mathbb{R}$, $b_+, b_- \in \mathbb{C}$, $\delta \in [0,1[$, P the bijective pseudodifferential operator*

$$P : H^{s+\beta}_{1-\text{per}}(\mathbb{R}) \to H^{s}_{1-\text{per}}(\mathbb{R}) : u \mapsto b_+ Q^\beta_+ u + b_- Q^\beta_- u + u_0$$

and assume that the conditions $s > \frac{1}{2}$, $r + \frac{1}{2} > s + \beta > \frac{1}{2}$ and $2m - s > \frac{1}{2}$ are satisfied. There is $c > 0$ such that

$$\sup_{g \in \mathcal{V}_j^{(2m)},\, \|g\|_{H^{2m-s}_{1-\text{per}}(\mathbb{R})} = 1} |\langle Pf, g\rangle_{H^m_{1-\text{per}}(\mathbb{R})}| \geq c \|f\|_{H^{s+\beta}_{1-\text{per}}(\mathbb{R})}$$

for every $j \in \mathbb{N}$ and every $f \in \mathcal{V}_{j,\delta}^{(r+1)}$, if and only if there is $c_1 > 0$ such that

$$\sup_{g \in \mathcal{V}_j^{(2m)},\, \|g\|_{2m-s} = 1} |\langle Pf, g\rangle_m| \geq c_1 \|f\|_{s+\beta}$$

for every $j \in \mathbb{N}$ and every $f \in \mathcal{V}_{j,\delta}^{(r+1)}$.

PROOF. Let us prove that the condition is necessary.

Using Proposition 3.37, one has merely to prove that there exist an operator K, compact from $H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})$ into itself and a constant $c' > 0$ such that

$$\sup_{g\in\mathcal{V}_j^{(2m)},\, \|g\|_{2m-s}=1} |\langle Pf\,,\,g\rangle_m| \geq c'\|f\|_{s+\beta} - \|Kf\|_{s+\beta}$$

for every $j \in \mathbb{N}$ and every $f \in \mathcal{V}_{j,\delta}^{(\tau+1)}$.

Let $j \in \mathbb{N}$ and $f \in \mathcal{V}_{j,\delta}^{(\tau+1)}$. For any $g \in \mathcal{V}_j^{(2m)}$, we have

$$|\langle Pf\,,\,g\rangle_m| \geq |\langle Pf\,,\,g\rangle_{H^m_{1-\mathrm{per}}(\mathbb{R})}| - |(Pf)(0)\,\overline{g(0)} - f_0\,\overline{g_0}\,|.$$

It follows that there is a constant $c' > 0$ such that

$$\sup_{g\in\mathcal{V}_j^{(2m)},\|g\|_{2m-s}=1} |\langle Pf\,,\,g\rangle_m| \geq c'\|f\|_{s+\beta} - \sup_{g\in\mathcal{V}_j^{(2m)},\|g\|_{2m-s}=1} |(Pf)(0)\,\overline{g(0)} - f_0\,\overline{g_0}\,|.$$

There is a constant $C > 0$ such that, for any $g \in \mathcal{V}_j^{(2m)}$ such that $\|g\|_{2m-s} = 1$, we have

$$\begin{aligned}
|(Pf)(0)\,\overline{g(0)} - f_0\,\overline{g_0}\,|^2 &\leq 2\,|(Pf)(0)|^2|g(0)|^2 + 2\,|f_0|^2|g_0|^2 \\
&\leq C\Big(|(Pf)(0)|^2 + |f_0|^2\Big).
\end{aligned}$$

Then we define the operator

$$K : H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R}) \to H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R}) \quad f \mapsto Kf$$

by

$$\begin{aligned}
(Kf)_0 &= \sqrt{C}\,(Pf)(0) \\
(Kf)_1 &= \sqrt{C}\,f_0 \\
(Kf)_k &= 0, \quad \forall k \in \mathbb{Z}\setminus\{0,1\}.
\end{aligned}$$

This operator K is compact from $H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})$ into $H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})$ since for every $f \in H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})$, we have

$$\begin{aligned}
\|Kf\|^2_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})} &\leq C\Big(2\,\Big|b_+\sum_{k\in\mathbb{Z}_0}|k|^\beta f_k + b_-\sum_{k\in\mathbb{Z}_0}\mathrm{sgn}(k)\,|k|^\beta f_k\Big|^2 + 3\,|f_0|^2\Big) \\
&= C\Big(2\,|\langle f\,,\,u\rangle_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})}|^2 + 3\,|\langle f\,,\,v\rangle_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})}|^2\Big)
\end{aligned}$$

where the functions $u, v \in H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})$ (since $s > \frac{1}{2}$) are defined by

$$u_0 := 0, \quad u_k := (\,\overline{b_+} + \overline{b_-}\ \mathrm{sgn}(k))|k|^{-2s-\beta},\ \forall k \in \mathbb{Z}_0$$

and

$$v_0 := 1, \quad v_k := 0,\ \forall k \in \mathbb{Z}_0.$$

We get

$$\sup_{g\in\mathcal{V}_j^{(2m)},\|g\|_{2m-s}=1} |(Pf)(0)\,\overline{g(0)} - f_0\,\overline{g_0}| \le \|Kf\|_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})},$$

which proves that the condition is necessary.

An easy adaptation of the proof above shows that the condition is sufficient. ∎

The next proposition gives some important particular cases of application. It is an obvious corollary of Theorem 3.41 based on the second and third points of Lemma 3.40.

Proposition 3.43 *1) Assume ($b_- = 0$ and $b_+ \neq 0$) or ($b_+, b_- \in \mathbb{R}$ and $\gamma > 0$). If r is odd (respectively r is even), the coercivity condition is satisfied if and only if $\delta \neq \frac{1}{2}$ (respectively $\delta \neq 0$).*
2) Assume $\gamma > 0$. If r is odd (respectively r is even), the coercivity condition is satisfied in case $\delta = 0$ (respectively $\delta = \frac{1}{2}$). ∎

Remark 3.44 *The condition $\gamma > 0$ is called the strong ellipticity condition (see [3]).*

Now in a second step, let us consider our more general operator

$$P = b_+Q_+^\beta + b_-Q_-^\beta + K_0.$$

As it is proved in Proposition 3.46, the coercivity conditions for the norms $\|\cdot\|_{H^s_{1-\mathrm{per}}(\mathbb{R})}$ and $\|\cdot\|_s$ can be deduced from Theorem 3.41. Lemma 3.45 will be useful to prove this proposition.

Lemma 3.45 *Let H_1, H_2 be Hilbert spaces and K_1 be a compact operator from H_1 into H_2. Then there is a compact operator K from H_1 into itself such that*

$$\|K_1u\|_{H_2} = \|Ku\|_{H_1}, \quad \forall u \in H_1.$$

PROOF. There are an orthonormal sequence $(u_m)_{m\in\mathbb{N}_0}$ of H_1 and a sequence $(\lambda_m)_{m\in\mathbb{N}_0}$ of $[0, +\infty[$ such that $\lambda_m \to 0$ if $m \to +\infty$ and

$$K_1^*K_1u = \sum_{m\in\mathbb{N}_0} \lambda_m \langle u\,,\,u_m\rangle_{H_1} u_m, \quad \forall u \in H_1.$$

The operator K defined by

$$Ku = \sum_{m\in\mathbb{N}_0} \sqrt{\lambda_m}\,\langle u\,,\,u_m\rangle_{H_1} u_m, \quad \forall u \in H_1$$

satisfies all the required conditions. ∎

Proposition 3.46 *Let m be a strictly positive integer, $r \in \mathbb{N}$, $s, \beta \in \mathbb{R}$, $b_+, b_- \in \mathbb{C}$, $\delta \in [0,1[$, P the bijective pseudodifferential operator*

$$P : H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R}) \to H^{s}_{1-\mathrm{per}}(\mathbb{R}) : \ u \mapsto b_+ Q^{\beta}_{+} u + b_- Q^{\beta}_{-} u + K_0$$

where for all $t \in \mathbb{R}$, K_0 is compact from $H^{t+\beta}_{1-\mathrm{per}}(\mathbb{R})$ into $H^{t}_{1-\mathrm{per}}(\mathbb{R})$. Assume that the conditions $s > \frac{1}{2}$ and $r + \frac{1}{2} > s + \beta$ are satisfied and that the principal part of P satisfies the inequality of Theorem 3.41. Then there are c_1, $c_2 > 0$ and compact operators K_1, $K_2 : H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R}) \to H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})$ such that

$$\sup_{g \in \mathcal{V}_j^{(2m)},\, \|g\|_{H^{2m-s}_{1-\mathrm{per}}(\mathbb{R})}=1} |\langle Pf \,,\, g\rangle_{H^m_{1-\mathrm{per}}(\mathbb{R})}| \geq c_1 \|f\|_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})} - \|K_1 f\|_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})}$$

and

$$\sup_{g \in \mathcal{V}_j^{(2m)},\, \|g\|_{2m-s}=1} |\langle Pf \,,\, g\rangle_m| \geq c_2 \|f\|_{s+\beta} - \|K_2 f\|_{s+\beta}$$

for every j and every $f \in \mathcal{V}^{(r+1)}_{j,\delta}$. These inequalities remain also valid with $K_1 = 0$, $K_2 = 0$ and smaller constants c_1, c_2.

Proof. Let us prove the first coercivity condition. The second one can be established in the same way.

We write $P = P_0 + K'$ with $K' := K_0 - T$ and $T : u \mapsto u_0$. Let $j \in \mathbb{N}_0$ and $f \in \mathcal{V}^{(r+1)}_{j,\delta}$. We have

$$\begin{aligned} &\sup_{g \in \mathcal{V}_j^{(2m)},\, \|g\|_{H^{2m-s}_{1-\mathrm{per}}(\mathbb{R})}=1} |\langle Pf \,,\, g\rangle_{H^m_{1-\mathrm{per}}(\mathbb{R})}| \\ &\geq \sup_{g \in \mathcal{V}_j^{(2m)},\, \|g\|_{H^{2m-s}_{1-\mathrm{per}}(\mathbb{R})}=1} |\langle P_0 f \,,\, g\rangle_{H^m_{1-\mathrm{per}}(\mathbb{R})}| - \sup_{g \in \mathcal{V}_j^{(2m)},\, \|g\|_{H^{2m-s}_{1-\mathrm{per}}(\mathbb{R})}=1} |\langle K' f \,,\, g\rangle_{H^m_{1-\mathrm{per}}(\mathbb{R})}| \\ &\geq c_1 \|f\|_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})} - \|K' f\|_{H^{s}_{1-\mathrm{per}}(\mathbb{R})}. \end{aligned}$$

The operator K' is compact from $H^{t+\beta}_{1-\mathrm{per}}(\mathbb{R})$ into $H^{t}_{1-\mathrm{per}}(\mathbb{R})$, for every $t \in \mathbb{R}$. By Lemma 3.45, there is a operator K'' compact from $H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})$ into itself such that

$$\|K'' u\|_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})} \leq \|K' u\|_{H^{s}_{1-\mathrm{per}}(\mathbb{R})}, \quad \forall u \in H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R}).$$

The last part is a direct consequence of Proposition 3.37. ∎

3.2.6 Direct consequence

Proposition 3.47 *We assume that the assumptions of Céa lemma are satisfied with*

$$X = H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R}), \quad Y = H^{s}_{1-\mathrm{per}}(\mathbb{R})$$

$$V_j = \mathcal{V}^{(r+1)}_{j,\delta}, \quad T_j = \{\langle \cdot \,,\, g\rangle_{H^m_{1-\mathrm{per}}(\mathbb{R})} : g \in \mathcal{V}_j^{(2m)}\}, \quad \forall j \in \mathbb{N}_0$$

($s \in \mathbb{R}$, $m \in \mathbb{N}_0$, $r \in \mathbb{N}$, $\delta \in [0,1[$). Then, $\forall j \in \mathbb{N}_0$, we have

$$\|u - u_j\|_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})} \leq C (2^{-j})^{S-(s+\beta)} \|u\|_{H^{S}_{1-\mathrm{per}}(\mathbb{R})}, \quad \forall u \in H^{S}_{1-\mathrm{per}}(\mathbb{R})$$

if $s + \beta \leq S \leq r + 1$ and $s + \beta < r + \frac{1}{2}$. ∎

3.2.7 Nitsche trick

Proposition 3.48 *We assume again that the assumptions of Céa lemma are satisfied with*

$$X = H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R}), \quad Y = H^{s}_{1-\mathrm{per}}(\mathbb{R})$$

$$V_j = \mathcal{V}^{(r+1)}_{j,\delta}, \quad T_j = \{\langle \cdot\,, g\rangle_{H^m_{1-\mathrm{per}}(\mathbb{R})} : g \in \mathcal{V}^{(2m)}_j\}, \quad \forall j \in \mathbb{N}_0$$

($s \in \mathbb{R}$, $m \in \mathbb{N}_0$, $r \in \mathbb{N}$, $\delta \in [0,1[$). If in addition, $P : H^{t+\beta}_{1-\mathrm{per}}(\mathbb{R}) \to H^{t}_{1-\mathrm{per}}(\mathbb{R})$ is an isomorphism $\forall t \in \mathbb{R}$ and $s > \frac{1}{2}$, then , $\forall j \in \mathbb{N}_0$, we have

$$\|u - u_j\|_{H^T_{1-\mathrm{per}}(\mathbb{R})} \leq C(T)(2^{-j})^{S-T}\|u\|_{H^S_{1-\mathrm{per}}(\mathbb{R})}, \quad \forall u \in H^S_{1-\mathrm{per}}(\mathbb{R})$$

if $\beta \leq T \leq s+\beta \leq S \leq r+1$ and $s+\beta < r+\frac{1}{2}$.

PROOF. Let $j \in \mathbb{N}_0$, $u \in H^S_{1-\mathrm{per}}(\mathbb{R})$ and $e := u - u_j$. We have $e \in H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R}) \subset H^{T}_{1-\mathrm{per}}(\mathbb{R})$. Since

$$\langle Pe\,, g_1\rangle_{H^m_{1-\mathrm{per}}(\mathbb{R})} = 0, \quad \forall g_1 \in \mathcal{V}^{(2m)}_j,$$

we get

$$\langle e\,, P^* g_1\rangle_{H^{m+\beta}_{1-\mathrm{per}}(\mathbb{R})} = 0, \quad \forall g_1 \in \mathcal{V}^{(2m)}_j.$$

We have

$$\|e\|_{H^T_{1-\mathrm{per}}(\mathbb{R})} = \sup_{\substack{h \in H^{2m+2\beta-T}_{1-\mathrm{per}}(\mathbb{R}),\\ \|h\|_{H^{2m+2\beta-T}_{1-\mathrm{per}}(\mathbb{R})}=1}} |\langle e\,, h\rangle_{H^{m+\beta}_{1-\mathrm{per}}(\mathbb{R})}|.$$

Let $h \in H^{2m+2\beta-T}_{1-\mathrm{per}}(\mathbb{R})$ such that $\|h\|_{H^{2m+2\beta-T}_{1-\mathrm{per}}(\mathbb{R})} = 1$. We have

$$\begin{aligned}
|\langle e\,, h\rangle_{H^{m+\beta}_{1-\mathrm{per}}(\mathbb{R})}| &= |\langle e\,, P^*g\rangle_{H^{m+\beta}_{1-\mathrm{per}}(\mathbb{R})}| \quad \text{with} \quad g \in H^{2m+\beta-T}_{1-\mathrm{per}}(\mathbb{R})\\
&= |\langle e\,, P^*(g-g_1)\rangle_{H^{m+\beta}_{1-\mathrm{per}}(\mathbb{R})}|, \quad \forall g_1 \in \mathcal{V}^{(2m)}_j,\\
&\leq \|e\|_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})}\|P^*(g-g_1)\|_{H^{2m+\beta-s}_{1-\mathrm{per}}(\mathbb{R})}\\
&\leq C\|e\|_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})}\|g-g_1\|_{H^{2m-s}_{1-\mathrm{per}}(\mathbb{R})},
\end{aligned}$$

where C is a constant independent of g_1. Therefore we get

$$\begin{aligned}
|\langle e\,, h\rangle_{H^{m+\beta}_{1-\mathrm{per}}(\mathbb{R})}| &\leq C\|e\|_{H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})} \inf_{g_1 \in \mathcal{V}^{(2m)}_j} \|g-g_1\|_{H^{2m-s}_{1-\mathrm{per}}(\mathbb{R})}\\
&\leq C'(T)(2^{-j})^{S-(s+\beta)}\|u\|_{H^S_{1-\mathrm{per}}(\mathbb{R})}(2^{-j})^{2m+\beta-T-(2m-s)}\|g\|_{H^{2m+\beta-T}_{1-\mathrm{per}}(\mathbb{R})}\\
&= C'(T)(2^{-j})^{S-T}\|u\|_{H^S_{1-\mathrm{per}}(\mathbb{R})}\|g\|_{H^{2m+\beta-T}_{1-\mathrm{per}}(\mathbb{R})}\\
&= C'(T)(2^{-j})^{S-T}\|u\|_{H^S_{1-\mathrm{per}}(\mathbb{R})}\|(P^*)^{-1}h\|_{H^{2m+\beta-T}_{1-\mathrm{per}}(\mathbb{R})}\\
&\leq C''(T)(2^{-j})^{S-T}\|u\|_{H^S_{1-\mathrm{per}}(\mathbb{R})}\|h\|_{H^{2m+2\beta-T}_{1-\mathrm{per}}(\mathbb{R})}.
\end{aligned}$$

Hence we get

$$\|e\|_{H^T_{1-\mathrm{per}}(\mathbb{R})} \leq C''(T)(2^{-j})^{S-T}\|u\|_{H^S_{1-\mathrm{per}}(\mathbb{R})}, \quad \forall u \in H^S_{1-\mathrm{per}}(\mathbb{R}).$$

∎

3.3 Collocation methods

We consider our operator

$$P = b_+ Q_+^\beta + b_- Q_-^\beta + K_0$$

bijective from $H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R})$ into $H^s_{1-\mathrm{per}}(\mathbb{R})$. Let $\delta \in [0,1[$. We assume again that $s > \frac{1}{2}$, $2m - s > \frac{1}{2}$ and $r + \frac{1}{2} > s + \beta > \frac{1}{2}$ and we suppose that the principal part of P satisfies the technical condition of coercivity with $\theta = 2\pi\delta$.

To get an approximate solution of $Pu = f$, one looks for $u_j \in \mathcal{V}^{(r+1)}_{j,\delta}$ such that the collocation equations

$$(Pu_j)(2^{-j}k) = f(2^{-j}k), \quad k = 0, \dots, 2^j - 1, \tag{3.1}$$

are satisfied.

Proposition 3.49 *For $j \in \mathbb{N}$ large enough, the collocation system 3.1 is equivalent to the Galerkin system*

$$\langle Pu_j , g\rangle_m = \langle f , g\rangle_m, \quad g \in \mathcal{V}^{(2m)}_j.$$

PROOF. See [8]. ∎

Now we look at the condition number of the matrix of this system. The integer numbers m and r are fixed respectively in $\mathbb{N}_0$ and $\mathbb{N}$; $\delta \in [0,1[$. We have to choose some test and trial functions.

a) Let us precise the test functions.

For every s fixed in $]-\infty, 2m - \frac{1}{2}[$, let $\{v_p^{(s,2m)} : p \in \mathbb{N}\}$ be a Riesz basis of $H^s_{1-\mathrm{per}}(\mathbb{R})$ constituted by spline functions of degree $2m-1$ such that for every $j \in \mathbb{N}$, the functions $v_p^{(s,2m)}$ belong to $\mathcal{V}^{(2m)}_j$ for $0 \le p < 2^j$. The Riesz bounds are denoted by $c_s^{(2m)}$ and $C_s^{(2m)}$ $(0 < c_s^{(2m)} \le C_s^{(2m)})$.

If s satisfies $\frac{1}{2} < s < 2m - \frac{1}{2}$, the functions

$$v_0^{(s,2m)} = 1$$

and

$$v_p^{(s,2m)} = 2^{a(m-s)} \Theta_{m;a,b}$$

if $p = 2^a + b$ is the dyadic decomposition of p $(a \in \mathbb{N};\ 0 \le b < 2^a)$ provide a first example of such test functions.

More generally, if $\ell \in \{0, \dots, m\}$ and if s satisfies the condition $\frac{1}{2} + \ell - m < s < \ell + m - \frac{1}{2}$, then the functions

$$v_0^{(s,2m)} = 1$$

and

$$v_p^{(s,2m)} = 2^{a(\ell-s)} \Psi^{(\ell)}_{m;a,b}$$

if $p = 2^a + b$ is the dyadic decomposition of p ($a \in \mathbb{N}$; $0 \leq b < 2^a$) can be used as test functions.

b) Now let us precise the trial functions.

For every s fixed in $\mathbb{R}$ satisfying $s < r + \frac{1}{2}$, let $\{u_q^{(s,r+1)} : q \in \mathbb{N}\}$ be a Riesz basis of $H^s_{1-\mathrm{per}}(\mathbb{R})$ constituted by spline functions of degree r. We ask that for every $j \in \mathbb{N}$, the functions $u_q^{(s,r+1)}$ belong to $\mathcal{V}_{j,\delta}^{(r+1)}$ for $0 \leq q < 2^j$. Let us denote the Riesz bounds by $d_s^{(r+1)}$ and $D_s^{(r+1)}$ $(0 < d_s^{(r+1)} \leq D_s^{(r+1)})$.

For example, if r is odd, we can choose the Riesz basis

$$\{1, 2^{a(\ell-s)}\Psi^{(\ell)}_{\frac{r+1}{2}:a,b}(\cdot - 2^{-a-1}\delta) : a \in \mathbb{N},\ 0 \leq b < 2^a\}$$

with ℓ fixed in $\{0, \dots, \frac{r+1}{2}\}$ such that $|s - \ell| < \frac{r}{2}$.

The next result shows that for the test and trial functions chosen above, the condition number of the collocation system is essentially determined by the Riesz constants.

Proposition 3.50 *Let $m \in \mathbb{N}_0$, $r \in \mathbb{N}$, $s, \beta \in \mathbb{R}$, $b_+, b_- \in \mathbb{C}$ such that $\frac{1}{2} < s < 2m - \frac{1}{2}$, $r + \frac{1}{2} > s + \beta > \frac{1}{2}$ and $\delta \in [0,1[$. If the operator*

$$P : H^{s+\beta}_{1-\mathrm{per}}(\mathbb{R}) \to H^s_{1-\mathrm{per}}(\mathbb{R}) : u \mapsto b_+Q^\beta_+ u + b_- Q^\beta_- u + K_0 u$$

is bijective and has a principal part which satisfies the conditions of Theorem 3.41 with $\theta = 2\pi\delta$, then there is $R > 0$ such that the matrices $P^{(j)}$ $(j \in \mathbb{N})$ of dimension 2^{j+1} defined by

$$P^{(j)} := (\langle P u_q^{(s+\beta,r+1)}, v_p^{(2m-s,2m)} \rangle_m)_{0 \leq p,q < 2^{j+1}}$$

satisfy

$$\eta(P^{(j)}) \leq R \frac{D_{s+\beta}^{(r+1)} C_{2m-s}^{(2m)}}{d_{s+\beta}^{(r+1)} c_{2m-s}^{(2m)}}$$

if j is large enough. Here, for any $j \in \mathbb{N}$, η denotes the condition number for the ℓ^2 norm

$$\eta(P^{(j)}) = \|P^{(j)}\| \, \|(P^{(j)})^{-1}\|$$

with

$$\|\cdot\| := \sup_{\|y\|_{\mathbb{C}^{2^{j+1}}}=1} \|\cdot y\|_{\mathbb{C}^{2^{j+1}}} = \sup_{\|y\|_{\mathbb{C}^{2^{j+1}}}=1} \ \sup_{\|x\|_{\mathbb{C}^{2^{j+1}}}=1} |\langle \cdot x, y \rangle_{\mathbb{C}^{2^{j+1}}}|.$$

PROOF. See [8]. ∎

Chapter 4

Numerical experiments

We have performed some numerical experiments with the double and the single layer potentials. Our first aim is to test the asymptotic convergence of the condition number of the stiffness matrix obtained using the $\Theta_{m;j,k}$ functions. We give also some examples of convergence of the solution.

In this chapter, we present the results we have obtained as well as the tools used to carry out these numerical experiments.

4.1 Explicit computation of the functions $\Psi_m^{(\ell)}$

In this section, we explain how to get the polynomials that define the spline functions $\Psi_m^{(\ell)}$ ($m \in \mathbb{N}_0$, $\ell \in \{0, \ldots, m\}$).

For every $k \in \mathbb{N}$ and $j \in \{0, \ldots, k\}$, we denote by $P[j,k,x]$ the polynomial $N_{k+1}|_{[j,j+1]}(x)$. Those polynomials $P[j,k,x]$ ($k \in \mathbb{N}$, $j \in \{0, \ldots, k\}$) can be explicitly computed by induction as follows

$$\begin{aligned}
P[0,0,x] &= 1, \\
P[0,r+1,x] &= \int_0^x dy\, P[0,r,x-y], \quad \forall r \in \mathbb{N}, \\
P[j,r+1,x] &= \int_{x-j}^1 dy\, P[j-1,r,x-y] + \int_0^{x-j} dy\, P[j,r,x-y], \\
&\qquad \forall r \in \mathbb{N}_0,\ \forall j \in \{1, \ldots, r\}, \\
P[r+1,r+1,x] &= \int_{x-(r+1)}^1 dy\, P[r,r,x-y], \quad \forall r \in \mathbb{N}.
\end{aligned}$$

For every $m \in \mathbb{N}_0$ and $n \in \{0, \ldots, 2(2m-1)-1\}$, we denote by $PCW[n,m,x,0]$ the polynomial $\psi_m|_{[\frac{n}{2},\frac{n+1}{2}]}(x)$. We have

$$PCW[n,m,x,0] = 2 \sum_{i=\sup\{-m,n-2m+1\}}^{\inf\{2m-2,n-m\}} c_{i+m+1} P[n-i-m, m-1, 2x-i-m]$$

where the coefficients c_k, $k \in \{1, \ldots, 3m-1\}$, are the coefficients of the polynomial

$$e^{i(3m-2)\xi} p_m(\xi)|_{e^{i\xi}=z}$$

taken in decreasing order (c_1 is the coefficient of z^{3m-2}).

For every $m \in \mathbb{N}_0$, $\ell \in \{1, \ldots, m\}$ and $n \in \{0, \ldots, 2(2m-1)-1\}$, we denote by $PCW[n, m, x, \ell]$ the polynomial $\psi_m^{(\ell)}|_{[\frac{n}{2}, \frac{n+1}{2}]}(x)$. An inductive way to compute these polynomials is the following

$$\begin{aligned} Fcn[n, m, x, \ell] &= \int_{\frac{n}{2}}^{x} PCW[n, m, s, \ell]\, ds \\ PCW[n, m, x, \ell] &= \sum_{u=0}^{n-1} Fcn[u, m, \frac{u+1}{2}, \ell - 1] + Fcn[n, m, x, \ell - 1]. \end{aligned}$$

Of course, we get

$$\theta_m|_{[\frac{n}{2}, \frac{n+1}{2}]}(x) = PCW[n, m, x, m], \quad \forall m \in \mathbb{N}_0, \ \forall n \in \{0, \ldots, 2(2m-1)-1\}.$$

Another more direct way to compute $\theta_m|_{[\frac{n}{2}, \frac{n+1}{2}]}$ is the following

$$\begin{aligned} PCW[n, m, x, m] &= \frac{1}{2^{2m-1}} \sum_{k=-m+1}^{m-1} (-1)^k P[m+k, 2m-1, m+k] \\ &\quad P[n-m-k+1, 2m-1, 2x-m-k+1]. \end{aligned}$$

The corresponding periodic polynomials

$$PCWper[n, m, x, \ell, j] := \Psi_{m;j}^{(\ell)}|_{[\frac{n}{2^{j+1}}, \frac{n+1}{2^{j+1}}]}(x),\ j \in \mathbb{N},\ n \in \{0, \ldots, 2^{j+1}-1\},\ \ell \in \{0, \ldots, m\}$$

are given by

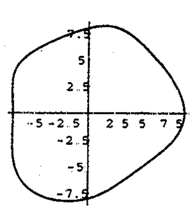
$$PCWper[n, m, x, \ell, j] = 2^{\frac{j}{2}-\ell j} \sum_{\mathrm{Ceiling}[2^{-j-1}(-4m+n+3)]}^{\mathrm{Floor}[2^{-j-1}n]} PCW[n - k2^{j+1}, m, 2^j(x-k), \ell]$$

where we have used Notation 4.1.

Notation 4.1 *If x is a real number, we denote by* Ceiling(x) *the smallest integer larger than x and by* Floor(x) *the largest integer smaller than x.*

4.2 Boundaries

We choose the two connected open subsets Ω of $\mathbb{R}^2$ represented on **Fig 12**.



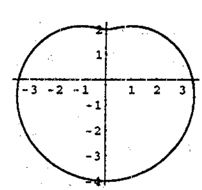


Fig 12

Their boundaries are smooth, connected and given respectively by

$$\begin{aligned}\gamma_1(t) &= \Big((\cos(6\pi t)+8)\cos(2\pi t), (\sin(4\pi t)+8)\sin(2\pi t)\Big), \quad t \in [0,1] \\ \gamma_2(t) &= \Big(3\sin(2\pi t)+\sin(4\pi t), -3\cos(2\pi t)-\cos(4\pi t)\Big), \quad t \in [0,1].\end{aligned}$$

We work with a parameterization by arc length. This requires to solve the autonomous differential equation

$$\mathrm{D}_s u = \frac{1}{\|\mathrm{D}\gamma(u)\|}, \quad u(0) = 0$$

where $\gamma(t)$, $t \in [0,1]$, is the parametrization of the boundary $\Gamma = \partial\Omega$. This can be done using a standard integration method (Runge-Kutta method of order 4) before the computation of the matrices and takes a very short time since it depends linearly on the number of points.

Let $\gamma^*(t)$ be the parametrization of Γ proportional to arc length and defined in $[0,1]$

$$\gamma^*(t) = \gamma(u(tL)), \quad L := \int_0^1 \|\mathrm{D}\gamma(t)\|\, dt.$$

4.3 The functions $\Theta_{m;i,k}$ as test and trial functions

In this section, we give a way to compute the elements of the matrices $P^{(j)}$ when the trial and test functions are chosen to be the same and are related to the functions $\Theta_{m;i,k}$.

If we work with the double or the single layer potential, we can remark that, as far as the test and trial bases are the same, the coercivity condition is satisfied for $\delta = 0$, independently of the choice of the spline functions.

Let $m \in \mathbb{N}_0$ and $j \in \mathbb{N}$. The matrix $P^{(j)}$ (see Proposition 3.50) is given by

$$P^{(j)} = \left(\langle P v_q^{(m+\frac{\beta}{2},2m)}, v_p^{(m+\frac{\beta}{2},2m)}\rangle_m\right)_{0\le p,q<2^{j+1}}$$

with

$$v_\ell^{(m+\frac{\beta}{2},2m)} = \begin{cases} 1 & \text{if } \ell = 0 \\ 2^{-i\frac{\beta}{2}}\Theta_{m;i,k} & \text{if } \ell = 2^i + k \in \mathbb{N}_0. \end{cases}$$

We get

$$P^{(j)} = (2\pi)^{-2m} L^{(j)} Q^{(j)}$$

where the matrices $L^{(j)}$ and $Q^{(j)}$ are defined by

$$\begin{aligned}(Q^{(j)})_{p,q} &= (P v_q^{(m+\frac{\beta}{2},2m)})(s_p), \quad 0 \le p,q < 2^{j+1} \\ (L^{(j)})_{p,q} &= (\mathrm{D}^{2m} v_p^{(m+\frac{\beta}{2},2m)})(s_q), \quad 0 < p < 2^{j+1},\ 0 \le q < 2^{j+1} \\ (L^{(j)})_{0,q} &= (2\pi)^{2m}\delta_{q,0}, \quad 0 \le q < 2^{j+1}.\end{aligned}$$

Here the collocation points s_ℓ are given by

$$s_\ell = \begin{cases} 0 & \text{if } \ell = 0 \\ \frac{1+2k}{2^{i+1}} & \text{if } \ell = 2^i + k. \end{cases}$$

The next two subsections explain how we have computed the matrices $Q^{(j)}$ and $L^{(j)}$.

4.3.1 Matrix $Q^{(j)}$

The computation of the elements of the matrix $Q^{(j)}$ can be performed easily using the Gauss-Legendre method with weights. It allows us to deal with a logarithm singularity in the case of the single layer potential.

More precisely, if we consider the single layer potential problem, the elements of the matrix $Q^{(j)}$ are computed as follows (j fixed in $\mathbb{N}$; $p \in \{1, \dots, 2^{j+1} - 1\}$, $q \in \{0, \dots, 2^{j+1} - 1\}$; $n := 2^{j+1} s_p$)

$$\begin{aligned} (Q^{(j)})_{p,q} &= (V v_q^{(m+\frac{\beta}{2},2m)})(s_p) \\ &= -\frac{L}{2\pi} \int_{2^{-j-1}(n+1)}^{2^{-j-1}(n-1)+1} \ln\left(|\gamma^*_{1-\text{per}}(s_p) - \gamma^*_{1-\text{per}}(s)|\right) v_{q,1-\text{per}}(s)\, ds \\ &\quad -\frac{L}{2\pi} \int_{2^{-j-1}(n-1)}^{2^{-j-1}(n+1)} \ln\left(\frac{|\gamma^*(s_p) - \gamma^*(s)|}{|s_p - s|}\right) v_q(s)\, ds \\ &\quad -\frac{L}{2\pi} \int_0^{2^{-j-1}} \ln(s)\ (v_q(s_p - s) + v_q(s_p + s))\, ds \end{aligned}$$

where for the sake of simplicity we write v_q instead of $v_q^{(m+\frac{\beta}{2},2m)}$ and with

$$\begin{aligned} \gamma^*_{1-\text{per}}(s) &:= \gamma^*(s \bmod(1)); \\ v_{q,1-\text{per}}(s) &:= v_q(s \bmod(1)), \quad \forall q \in \{0, \dots, 2^{j+1} - 1\}. \end{aligned}$$

The first integrals are evaluated using the classical Gauss-Legendre method for the weight $w(s) = 1$. The last one is evaluated by the same method for the weight $w(s) = \ln(s)$.

4.3.2 Matrix $L^{(j)}$

For every $i \in \mathbb{N}$ and $n \in \{0, \dots, 2^{i+1} - 1\}$, let the constant $C[n, m, i]$ be the coefficient of x^{m-1} in $PCWper[n, m, x, 0, i]$ and $C[-1, m, i] := C[2^{i+1} - 1, m, i]$. For every $x \in [0, 1[\setminus \cup_{n=0}^{2^{i+1}-1} \{\frac{n}{2^{i+1}}\}$, we have

$$D^{2m-1}\Theta_{m;i}(x) = (m-1)! \sum_{n=0}^{2^{i+1}-1} C[n, m, i]\ \chi_{]\frac{n}{2^{i+1}}, \frac{n+1}{2^{i+1}}[}(x).$$

Therefore, for every $i \in \{0, \dots, j\}$ and every $q \in \{0, \dots, 2^{j+1}-1\}$, we have

$$(L^{(j)})_{2^i,q} = 2^{-i\frac{\beta}{2}}(m-1)!\left(C[2^{i+1}s_q, m, i] - C[2^{i+1}s_q - 1, m, i]\right).$$

and, $\forall k \in \{1, \dots, 2^i - 1\}$, we get

$$(L^{(j)})_{2^i+k,q} = (L^{(j)})_{2^i,r}$$

if r is such that $s_r = (s_q - 2^{-i}k) \bmod (1)$. So, it is important to compute the s_ℓ and such r.

First of all, the collocation points s_ℓ ($\ell \in \{0, \dots, 2^{j+1}-1\}$) can be obtained by the following algorithm

$$\begin{array}{l} s_0 = 0; \\ \text{for}(i = 0;\ i \leq j;\ i+ = 1) \\ \quad \text{for}(k = 2^i, n = 1;\ k \leq 2^{i+1} - 1;\ k+ = 1, n+ = 2) \\ \quad\quad s_k = 2^{-i-1}n. \end{array}$$

For every $r \in \{1, \dots, j\}$, we denote by $p^{(r)} : \ell \mapsto p_\ell^{(r)}$ the inverse of the function $2^{r+1}s$ with

$$s: \{0, 1, \dots, 2^{r+1}-1\} \to \{0\} \cup \{2^{-i}n : i \in \{1, \dots, r+1\},\ n \in \{1, 3, \dots, 2^i - 1\}\} : \ell \mapsto s_\ell.$$

The points $p_\ell^{(r)}$ ($\ell \in \{0, \dots, 2^{r+1}-1\}$) can be obtained by the algorithm

$$\begin{array}{l} p_0^{(r)} = 0; \\ \text{for}(i = 0;\ i \leq r;\ i+ = 1) \\ \quad \text{for}(k = 2^i, n = 1;\ k \leq 2^{i+1} - 1;\ k+ = 1, n+ = 2) \\ \quad\quad p_{n2^{r-i}}^{(r)} = k. \end{array}$$

Now we can easily compute the inferior bloc triangular matrix $L^{(j)}$. The dimensions of the diagonal blocs are given by 1, 2^0, 2^1, 2^2, $\dots 2^j$. Remark that if $j \in \mathbb{N}_0$, we get

$$L^{(j)} = \left(\begin{array}{c|c} L^{(j-1)} & 0 \\ \hline A & B \end{array}\right)$$

for some matrices A, B. So, we initialize $L^{(j)}$ to 0 and fill in it level by level (by *a level* we mean all the lines corresponding to a diagonal bloc as presented above).

For every integer $r \in [J_m, j]$, we can fill in the $(r+2)$-th level as follows.

In a first step, we fill in the non-diagonal blocs using the algorithm

$$\begin{array}{l} \text{for}(i = 2^r, \ell = 0;\ i < 2^{r+1};\ i+ = 1, \ell+ = 1) \\ \quad \text{for}(n = 0;\ n < 2^r;\ n+ = 1) \\ \quad\quad \text{if}(\ell + n < 2^r)\ (L^{(j)})_{i, p_{\ell+n}^{(r-1)}} = 2^{-r\frac{\beta}{2}}(m-1)!\left(C[2n, m, r] - C[2n-1, m, r]\right) \\ \quad\quad \text{else}\ (L^{(j)})_{i, p_{\ell+n-2^r}^{(r-1)}} = 2^{-r\frac{\beta}{2}}(m-1)!\left(C[2n, m, r] - C[2n-1, m, r]\right). \end{array}$$

In a second step, the diagonal bloc is filled in in a more simply way using the algorithm

for$(i = 2^r, \ell = 0;\ i < 2^{r+1};\ i+ = 1, \ell+ = 1)$
for$(n = 0;\ n < 2^r;\ n+ = 1)$
if$(\ell + n < 2^r)$ $(L^{(j)})_{i,2^r+\ell+n} = 2^{-r\frac{\beta}{2}}(m-1)!\Big(C[2n+1,m,r] - C[2n,m,r]\Big)$
else $(L^{(j)})_{i,\ell+n} = 2^{-r\frac{\beta}{2}}(m-1)!\Big(C[2n+1,m,r] - C[2n,m,r]\Big).$

4.3.3 Results

The figure "**Fig. 13**" (resp. "**Fig. 14**") gives the ℓ^2-condition number η of the matrix $P^{(j)}$ for the double and the single layer potential on γ_1 (resp. γ_2). It also gives the L^2 norm of the error (ε_j) for the second member $f(x) = 2\sin(2\pi x)$ and the estimated exponent of convergence

$$\text{eoc}(j) = \frac{\ln(\frac{\varepsilon_j}{\varepsilon_{j+1}})}{\ln(2)}.$$

Remind that the trial and test functions are chosen to be the same $(s+\beta = 2m-s)$.

For the double layer potential, we have therefore $m = s$ and for $m > \frac{1}{2}$, we can use the Riesz basis $\{1, \Theta_{m;j,k} : j \geq 0,\ 0 \leq k < 2^j\}$. We treat the cases $m = 1$ (linear splines), and $m = 2$ (cubic splines).

For the single layer potential, we have $m = s - \frac{1}{2}$ and for $m > 1$, we can use the Riesz basis $\{1, 2^{\frac{j}{2}}\Theta_{m;j,k} : j \geq 0,\ 0 \leq k < 2^j\}$. We can not use here linear splines.

	Double layer potential						Single layer potential		
j	$m=1$			$m=2$			$m=2$		
	η	error	eoc	η	error	eoc	η	error	eoc
1	2.27	7.1 e-01		13.46	2.6 e-01		20.15	4.4 e-02	
2	2.37	2.0 e-01	1.82	15.85	9.0 e-02	1.55	203.12	4.1 e-02	0.10
3	2.42	5.2 e-02	1.96	18.12	6.0 e-03	3.91	535.43	6.8 e-03	2.58
4	2.44	1.3 e-02	1.95	18.25	5.4 e-04	3.47	793.66	7.1 e-04	3.27
5	2.44	3.3 e-03	2.01	18.27	1.9 e-05	4.83	831.60	2.0 e-05	5.16
6	2.45	8.0 e-04	2.06	18.27	8.3 e-07	4.52	833.82	7.6 e-07	4.70
7	2.45	1.7 e-04	2.27	18.27	4.0 e-08	4.37	834.07	4.3 e-08	4.13

Fig 13

	Double layer potential						Single layer potential		
j	$m=1$			$m=2$			$m=2$		
	η	error	eoc	η	error	eoc	η	error	eoc
1	2.35	5.2 e-01		12.87	8.7 e-02		23.69	3.1 e-01	
2	2.40	1.5 e-01	1.80	13.76	4.5 e-02	0.96	163.97	1.9 e-01	0.73
3	2.41	4.2 e-02	1.83	14.35	1.5 e-02	1.62	565.02	1.3 e-01	0.48
4	2.41	1.1 e-02	1.96	15.32	2.9 e-03	2.35	921.16	8.1 e-02	0.73
5	2.42	2.7 e-03	2.04	15.92	2.6 e-04	3.43	982.99	2.2 e-02	1.89
6	2.42	6.4 e-04	2.06	16.00	8.4 e-06	4.96	989.04	1.3 e-03	4.05
7	2.43	1.4 e-04	2.21	16.00	3.6 e-07	4.56	990.20	3.1 e-05	5.42

Fig 14

4.4 A comparison with the functions $\Psi_{2m;i,k}$

For m fixed in $\mathbb{N}_0$ and $j \in \mathbb{N}$, it seems natural to compare the construction that we present in the previous section ($\Theta_{m;i,k}$ as test and trial functions) with the analogous construction using the functions $\Psi_{2m;i,k}$.

Let us expose briefly how to compute the matrix

$$P^{(j)} = \left(\langle P v_q^{(m+\frac{\beta}{2},2m)}, v_p^{(m+\frac{\beta}{2},2m)}\rangle_m\right)_{0\le p,q<2^{j+1}}$$

if

$$v_\ell^{(m+\frac{\beta}{2},2m)} = \begin{cases} 1 & \text{if } \ell = 0 \\ 2^{-(m+\frac{\beta}{2})i}\Psi_{2m;i,k} & \text{if } \ell = 2^i + k \in \mathbb{N}_0. \end{cases}$$

Retaining the definition of the collocation points given in the previous section, we get

$$P^{(j)} = Y^{(j)}\,\widetilde{L_0}^{(j)} + (2\pi)^{-2m}\mathcal{L}^{(j)}Q^{(j)}$$

where the matrices $L^{(j)}$ and $Q^{(j)}$ are defined by

$$\begin{aligned} (Q^{(j)})_{p,q} &= (P v_q^{(m+\frac{\beta}{2},2m)})(s_p), \quad 0 \le p,q < 2^{j+1} \\ (\mathcal{L}^{(j)})_{p,q} &= (\mathrm{D}^{2m} v_p^{(m+\frac{\beta}{2},2m)})(s_q), \quad 0 \le p,q < 2^{j+1}, \end{aligned}$$

and the vectors $Y^{(j)}$ and $L_0^{(j)}$ by

$$\begin{aligned} (Y^{(j)})_p &= v_p^{(m+\frac{\beta}{2},2m)}(0), \quad 0 \le p < 2^{j+1} \\ (L_0^{(j)})_q &= (Q^{(j)})_{0,q}, \quad 0 \le q < 2^{j+1}. \end{aligned}$$

The chart "**Fig. 15**" gives a comparison between the condition numbers obtained with the functions $\Theta_{m;i,k}$ and $\Psi_{2m;i,k}$. This comparison is carried out in the case of the double layer potential on γ_1.

j	$m=1$		$m=2$		$m=3$	
	Fcts $\Theta_{1;i,k}$	Fcts $\Psi_{2;i,k}$	Fcts $\Theta_{2;i,k}$	Fcts $\Psi_{4;i,k}$	Fcts $\Theta_{3;i,k}$	Fcts $\Psi_{6;i,k}$
3	2.42	21.94	18.12	87.98	150.22	797.11
4	2.44	22.84	18.25	88.03	150.50	797.13
5	2.44	23.58	18.27	88.05	150.58	797.13
6	2.45	24.18	18.27	88.06	150.58	797.13
7	2.45	24.68	18.27	88.06	150.58	797.13

Fig. 15

4.5 Trial basis different from the test basis

In this section, we present three examples of stiffness matrices in which test and trial functions are chosen to be different. They are all concerned with the double layer potential on γ_1

4.5.0.1 Example 1: $s = m = 1;\ r = 3$

We choose as bases

$$\begin{aligned} u_q &= \begin{cases} 1 & \text{if } q = 0 \\ 2^a\Theta_{2;a,b} & \text{if } q = 2^a + b \in \mathbb{N}_0, \end{cases} \\ v_p &= \begin{cases} 1 & \text{if } p = 0 \\ \Theta_{1;a,b} & \text{if } p = 2^a + b \in \mathbb{N}_0 \end{cases} \end{aligned}$$

and we get

$$P^{(j)} = (\langle Pu_q\,,\, v_p\rangle_1)_{0\le p,q<2^{j+1}} = L^{(j)}Q^{(j)}$$

where $L^{(j)}$ is the matrix $L^{(j)}$ previously given for $m = 1$ and where

$$(Q^{(j)})_{p,q} = (Pu_q)(s_p), \quad 0 \le p, q < 2^{j+1}.$$

4.5.0.2 Example 2: $s = m = 2;\ r = 5$

We choose as bases

$$\begin{aligned} u_q &= \begin{cases} 1 & \text{if } q = 0 \\ 2^a\Theta_{3;a,b} & \text{if } q = 2^a + b \in \mathbb{N}_0, \end{cases} \\ v_p &= \begin{cases} 1 & \text{if } p = 0 \\ \Theta_{2;a,b} & \text{if } p = 2^a + b \in \mathbb{N}_0 \end{cases} \end{aligned}$$

and we get

$$P^{(j)} = (\langle Pu_q\,,\, v_p\rangle_2)_{0\le p,q<2^{j+1}} = L^{(j)}Q^{(j)}$$

where $L^{(j)}$ is the matrix $L^{(j)}$ previously given for $m = 2$ and where

$$(Q^{(j)})_{p,q} = (Pu_q)(s_p), \quad 0 \le p, q < 2^{j+1}.$$

4.5.0.3 Example 3: $s = m = 3;\ r = 3$

Here, we choose as bases

$$u_q = \begin{cases} 1 & \text{if } q = 0 \\ 2^{-a}\Theta_{2;a,b} & \text{if } q = 2^a + b \in \mathbb{N}_0, \end{cases}$$
$$v_p = \begin{cases} 1 & \text{if } p = 0 \\ \Theta_{3;a,b} & \text{if } p = 2^a + b \in \mathbb{N}_0 \end{cases}$$

and we get

$$P^{(j)} = (\langle Pu_q\,,\, v_p\rangle_3)_{0\le p,q<2^{j+1}} = L^{(j)}Q^{(j)}$$

where $L^{(j)}$ is the matrix $L^{(j)}$ previously given for $m = 3$ and where

$$(Q^{(j)})_{p,q} = (Pu_q)(s_p), \quad 0 \le p, q < 2^{j+1}.$$

4.5.0.4 Condition number

	Double layer potential								
j	Example 1			Example 2			Example 3		
	η	error	eoc	η	error	eoc	η	error	eoc
1	4.51	6.7 e-02		5.96	6.2 e-02		7.22	1.2 e-00	
2	4.95	1.3 e-02	2.34	7.00	1.3 e-02	2.24	19.07	7.3 e-01	0.74
3	5.06	4.0 e-04	5.03	8.03	4.3 e-04	4.92	38.20	9.9 e-02	2.88
4	5.20	1.9 e-05	4.42	8.11	1.3 e-05	5.05	47.17	1.8 e-02	2.48
5	5.32	5.2 e-07	5.18	8.11	8.1 e-08	7.34	49.64	1.2 e-03	3.82
6	5.42	2.6 e-08	4.35	8.11	2.1 e-10	8.55	49.87	1.1 e-04	3.52
7	5.51	1.4 e-09	4.16	8.11	8.1 e-12	4.72	49.92	1.0 e-05	3.42

Fig. 16

4.6 A remark about the first basis functions

When we have a look at the figures "**Fig. 13**" and "**Fig. 14**", we see that in the single layer potential case, the condition number grows up significantly during the first levels. So it turns out that it is important to choose very carefully the first functions whose support is the full interval. This concerns only a few number of functions ($2^{\text{Ceiling}(J_m)}$ for splines of degree $2m - 1$) and can significantly modify the results. This is quite natural since the choice of the basis is made to obtain a very good asymptotic behavior which can be damaged by a bad choice of the first functions.

So we propose in this section some alternative first functions. They can in some cases lead to a significantly lower condition number. For example, in the single layer potential case on the boundary γ_1, the condition number of the stiffness matrix decreases from 831 down to 34 when we use the first functions presented below with a convenient constant factor.

4.6.1 General setting

Let m fixed in $\mathbb{N}_0$ and let ℓ_m be the integer Ceiling(J_m). We define the functions S_j, $j \in \{0, \ldots, 2^{\ell_m} - 1\}$, by

$$\begin{aligned} S_0(x) &:= 1 \\ S_j(x) &:= \sum_{k\in\mathbb{Z}} \frac{e^{2i\pi(2^{\ell_m}k+j)x}}{(2^{\ell_m}k+j)^{2m}}, \quad j \in \{1, \ldots, 2^{\ell_m} - 1\}. \end{aligned}$$

The functions $S_0, \ldots, S_{2^{\ell_m}-1}$ are orthogonal in $L^2(]0,1[)$ and form a basis of $\mathcal{V}_{\ell_m}^{(2m)}$.

For every $j \in \{0, \ldots, 2^{\ell_m} - 1\}$, one gets

$$S_j(x - 2^{-\ell_m}) = e^{-2^{1-\ell_m}ij\pi} S_j(x),$$

so it suffices to compute each function S_j on $I :=]-2^{-\ell_m-1}, 2^{-\ell_m-1}[$.

For $j \in \{1, \ldots, 2^{\ell_m} - 1\}$ and $x \in I$, we have

$$S_j(x) = -e^{2ij\pi x}\,\mathrm{Res}_{-2^{-\ell_m}j}\Big(\frac{\pi}{\sin(\pi z)}\, f_x^{(j)}(z)\Big)$$

with

$$f_x^{(j)}(z) := \frac{e^{iz(\theta-\mathrm{sgn}(\theta)\pi)}}{(2^{\ell_m}z+j)^{2m}}, \quad \theta := 2^{\ell_m+1}\pi x.$$

4.6.2 Explicit computation of those functions for $m = 2$

The functions S_j $(j = 1, 2, 3)$ are given on $]\frac{1}{8}, \frac{1}{8}[$ by

$$\begin{aligned} S_1(x) &= \frac{\pi^4}{96}(1 + 6ix - 24x^2 + 32(\mathrm{sgn}(x) - i)x^3) \\ S_2(x) &= \frac{\pi^4}{768}(1 - 96x^2 + 256\,\mathrm{sgn}(x)x^3) \\ S_3(x) &= \overline{S_1(x)}. \end{aligned}$$

The use of the first functions

$$u_0 := \frac{S_0}{\|S_0\|_{L^2(]0,1[)}},\ u_1 := \frac{\mathcal{R}e(S_1)}{\|\mathcal{R}e(S_1)\|_{L^2(]0,1[)}},\ u_2 := \frac{S_2}{\|S_2\|_{L^2(]0,1[)}},\ u_3 := \frac{\mathcal{I}m(S_1)}{\|\mathcal{I}m(S_1)\|_{L^2(]0,1[)}}$$

which are orthonormal in $L^2(]0,1[)$ give good results. On $]0,1[$, they are given explicitly by

$$\begin{aligned} u_0(x) &= 1 \\ u_1(x) &= \sqrt{\frac{35}{17}}\Big(1 - 24x^2 + 32x^3 - 8(2x-1)_+^3\Big) \\ u_2(x) &= \sqrt{\frac{35}{17}}\Big(1 - 96x^2 + 256x^3 + 8\sum_{k=1}^{3}(-1)^k(4x-k)_+^3\Big) \\ u_3(x) &= \sqrt{\frac{35}{17}}\Big(6x - 32x^3 + \sum_{k=0}^{1}(-1)^k(4x-1-2k)_+^3\Big). \end{aligned}$$

The figure Fig. 17 gives a representation of those functions.

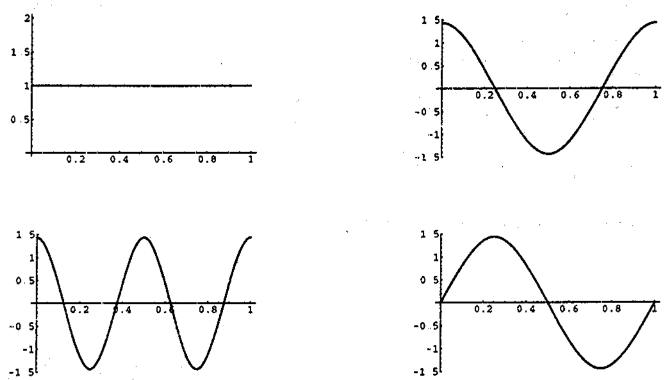


Fig. 17

4.6.3 Explicit computation of those functions for $m = 3$

The functions

$$u_0 = 1, \quad u_1 = \frac{S_4}{\|S_4\|_{L^2(]0,1[)}}, \quad u_2 = \frac{\mathcal{R}e(S_2)}{\|\mathcal{R}e(S_2)\|_{L^2(]0,1[)}}, \quad u_3 = \frac{\mathcal{I}m(S_2)}{\|\mathcal{I}m(S_2)\|_{L^2(]0,1[)}},$$
$$u_4 = \frac{\mathcal{R}e(S_1)}{\|\mathcal{R}e(S_1)\|_{L^2(]0,1[)}}, \quad u_5 = \frac{\mathcal{I}m(S_1)}{\|\mathcal{I}m(S_1)\|_{L^2(]0,1[)}}, \quad u_6 = \frac{\mathcal{R}e(S_3)}{\|\mathcal{R}e(S_3)\|_{L^2(]0,1[)}},$$
$$u_7 = \frac{\mathcal{I}m(S_3)}{\|\mathcal{I}m(S_3)\|_{L^2(]0,1[)}},$$

are orthonormal in $L^2(]0,1[)$. They are given explicitly on $]0,1[$ by

$$\begin{aligned}
u_0(x) &= 1\\
u_1(x) &= \sqrt{\frac{1386}{691}}\left(1 - 320x^2 + 20480x^4 - 65536x^5 - 4\sum_{k=1}^{7}(-1)^k(8x-k)_+^5\right)\\
u_2(x) &= \sqrt{\frac{693}{1382}}\left(2 - 160x^2 + 2560x^4 - 4096x^5 - 8\sum_{k=1}^{3}(-1)^k(4x-k)_+^5\right)\\
u_3(x) &= \sqrt{\frac{693}{1382}}\left(25x - 640x^3 + 4096x^5 - \frac{1}{4}\sum_{k=0}^{3}(-1)^k(8x-1-2k)_+^5\right)\\
u_4(x) &= \sqrt{\frac{2772}{3930044 + 2396369\sqrt{2}}}\left(-26 - 33\sqrt{2} + 640(-2+3\sqrt{2})x^2 - 20480(-4+3\sqrt{2})x^4\right.\\
&\quad \left. +(10-7\sqrt{2})\left(\sum_{k=0}^{1}(-1)^k(8x-1-2k)_+^5 - \sum_{k=2}^{3}(-1)^k(8x-1-2k)_+^5\right) + 32768(-7+5\sqrt{2})x^5 - 2048(-7+5\sqrt{2})(2x-1)_+^5\right)\\
u_5(x) &= \sqrt{\frac{2772}{3930044 + 2396369\sqrt{2}}}\left(-40(10+\sqrt{2})x - 5120(-2+\sqrt{2})x^3 + 32768(-3+2\sqrt{2})x^5\right.\\
&\quad \left. +(10-7\sqrt{2})\left(\sum_{k=0}^{1}(8x-1-2k)_+^5 - \sum_{k=2}^{3}(8x-1-2k)_+^5\right) + 64(-7+5\sqrt{2})(4x-1)_+^5\right)
\end{aligned}$$

$$u_6(x) = \sqrt{-\frac{44352}{3930044 - 2396369\sqrt{2}}}\left(\frac{-26+33\sqrt{2}}{4} - 160(2+3\sqrt{2})x^2 + 5120(4+3\sqrt{2})x^4\right.$$

$$\left. + \frac{10+7\sqrt{2}}{4}\left(\sum_{k=0}^{1}(-1)^k(8x-1-2k)_+^5 - \sum_{k=2}^{3}(-1)^k(8x-1-2k)_+^5\right) - 8192(7+5\sqrt{2})x^5 + 512(7+5\sqrt{2})(2x-1)_+^5\right)$$

$$u_7(x) = \sqrt{\frac{44352}{3930044 - 2396369\sqrt{2}}}\left(10(10-\sqrt{2})x - 1280(2+\sqrt{2})x^3 + 8192(3+2\sqrt{2})x^5\right.$$

$$-\frac{10+7\sqrt{2}}{4}\left(\sum_{k=0}^{1}(8x-1-2k)_+^5 - \sum_{k=2}^{3}(8x-1-2k)_+^5\right)$$

$$\left. +16(7+5\sqrt{2})\sum_{k=0}^{1}(-1)^k(4x-1-2k)_+^5\right)$$

Those functions are represented on **Fig. 18**.

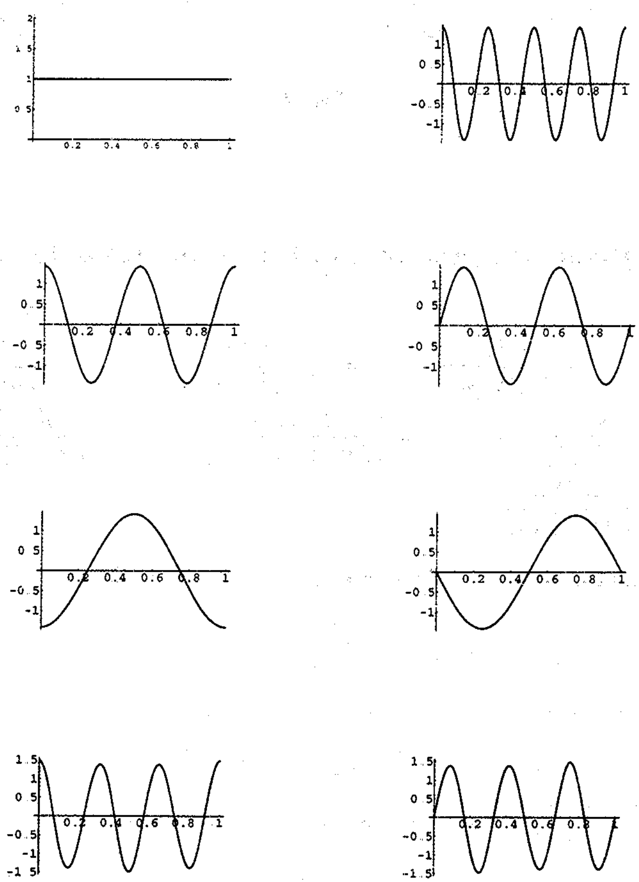


Fig. 18

4.6.4 Examples

Here you have a comparison between some condition numbers presented in "**Fig. 13**" and "**Fig. 14**" (left columns) and the one obtained using the alternative functions (right columns). It concerns cubic splines.

	Single layer potential			
j	On γ_1		On γ_2	
1	20.15	5.95	23.69	3.40
2	203.12	25.44	163.97	24.67
3	535.43	27.26	565.02	27.53
4	793.66	29.88	921.16	29.11
5	831.60	33.79	982.99	30.13
6	833.82	41.65	989.04	31.09
7	834.07	54.67	990.20	37.19

Fig. 19

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