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A FRÉCHET SPACE OF CONTINUOUS FUNCTIONS WHICH IS A PREQUOJECTION

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Abstract. The purpose of this paper is to show that the prequojection constructed by Moscatelli in [5, § 2] is quite a concrete space, namely, it is the space $C_{a,0}(K,\tau)$ of functions which are continuous and affine on a countable k-space (K,τ) and vanish at $0 \in K$.

Introduction. We recall that a *prequojection* is a Fréchet space F whose strong bidual F'' is a quojection and F is non-trivial if it is not itself a quojection. We also recall briefly the construction in [5]. Consider the duality $\langle c_0, l^1\rangle$. For all $m \geq 0$ put $l^1_m = l^1$ and then write l^1 as $\left(\bigoplus_m l^1_m\right)_1$. Let $(f^m_n)_{n\geq 0}$ be the standard basis of l^1_m, let s be a mapping of the non-negative integers onto themselves such that $s^{-1}(j)$ is infinite for all j and let (ε_n) be a sequence of positive numbers such that $1 > \varepsilon \geq \varepsilon_n \longrightarrow 0$. Consider the subspace M of l^1 defined by

$$M = \left[f^0_n + \varepsilon_{s(n)} f^1_{s(n)} + \varepsilon_{s(n)}\varepsilon_{s^2(n)} f^2_{s^2(n)} + \ldots : n \geq 0\right].$$

Put $M^0 = M$, let M^1 be the set of all limits of w^*-convergent and bounded nets in M^0 and, inductively, let $M^m = (M^{m-1})^1$ for $m > 1$. By Theorem 3 of [5] $M^m \neq l^1$

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for all m and hence M is *strongly non-norming*, i.e. no M^m is norming on l^1. For each $m \geq 0$ the polar in c_0 of the unit ball of $\overline{M^m}$ generates a norm $|\cdot|_m$ on c_0 and we denote by F_m the completion of $(c_0, |\cdot|_m)$, so that $F_m' = M^{m+1}$. Since there are natural maps $i_m : F_{m+1} \longrightarrow F_m$, the projective limit of the sequence (F_m) is a Fréchet space F which is a non-trivial prequojection and F is countably normed in the representation (F_m, i_m), i.e. the maps i_m are injective (cf. [2]).

It is our aim here to show that F is a classical space (indeed, a space of continuous functions) and this result is all the more remarkable in so far as F, like all prequojections, is known not to have the bounded approximation property (cf. [5]).

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1. F is the projective limit of spaces of continuous functions.

For all $m, j \geq 0$ put $g_j^m = f_j^{m+1} + \varepsilon_{s(j)} f_{s(j)}^{m+2} + \ldots$ and

$$K_m = (f_n^0) \cup \ldots (f_n^{m-1}) \cup \left(f_n^m + \varepsilon_{s(n)} g_{s(n)}^m\right) \cup \left(\varepsilon_j g_j^m\right) \cup (0). \tag{1}$$

Then K_m is a countable w^*-compact subset of l^1 which will always be endowed with the induced w^*-topology. Denoting by $C_0(K_m)$ the Banach space of continuous functions on K_m vanishing at 0, we have

Theorem 1.1. *$F_m \simeq C_0(K_m)$ for all $m \geq 0$.*

Proof. It is shown in the proof of Theorem 3 of [5] that $\overline{M^1} = l_0^1 \oplus [g_j^0]$ and, in the same way, it is easy to see that, with $l_i^1 \equiv l^1$ for $i = 1, \ldots, m$,

$$\overline{M^{m+1}} = \left(\bigoplus_{i=0}^{m} l_i^1\right) \oplus \left[g_j^m\right] \qquad \text{for all } m. \tag{2}$$

For a fixed $m \geq 0$ put

$$\lambda_m(x) = \sup\left\{|(x, h)| : h \in K_m\right\}, \qquad x \in F_m.$$

Let $x \in F_m$. If $x' \in F_m' = M^{m+1}$ with $\|x'\| \leq 1$, then there exists a sequence $(u_l) \subset M^m \subset \overline{M^m}$ such that $\|u_l\| \leq 1$ and $u_l \xrightarrow{w^*} x'$. By (2) we may write

$$u_l = \sum_{i=0}^{m-1} \sum_{n=0}^{k_l^i} a_{ln}^i f_n^i + \sum_{n=0}^{k_l^m} a_{ln}^m g_n^{m-1},$$

hence

$$\begin{aligned}|\langle x, u_l\rangle| &\le \sum_{i=0}^{m-1}\sum_{n=0}^{k_l^i}|a_{ln}^i|\,|\langle x, f_n^i\rangle| + \sum_{n=0}^{k_l^m}|a_{ln}^m|\left|\left\langle x, f_n^m+\varepsilon_{s(n)}g_{s(n)}^m\right\rangle\right|\\ &\le \lambda_m(x)\left(\sum_{i=0}^{m-1}\left\|\sum_{n=0}^{k_l^i}a_{ln}^i f_n^i\right\| + \left\|\sum_{n=0}^{k_l^m}a_{ln}^m\left(f_n^m+\varepsilon_{s(n)}g_{s(n)}^m\right)\right\|\right)\\ &\le c_m\lambda_m(x)\end{aligned}$$

for a suitable constant c_m. Letting $l \longrightarrow \infty$ we obtain

$$|\langle x, x'\rangle| \le c_m\lambda_m(x),$$

from which it follows that

(3) $$|x|_m \le c_m\lambda_m(x).$$

On the other hand,

$$|\langle x, f_n^i\rangle| \le |x|_m, \qquad i = 0, 1, \ldots, m-1,$$

and

$$\left|\left\langle x, f_n^m+\varepsilon_{s(n)}g_{s(n)}^m\right\rangle\right| \le \frac{1}{1-\varepsilon}|x|_m,$$

showing that

(4) $$\lambda_m(x) \le \frac{1}{1-\varepsilon}|x|_m.$$

Combining (3) and (4) we see that the map $T_m\colon F_m \longrightarrow C_0(K_m)$ given by $T_m x = x_{|K_m}$ is an isomorphism into. The proof will be complete if we show that $T_m(F_m)$ is dense in $C_0(K_m)$. To this end put

(5) $$K^m = \left(f_n^m+\varepsilon_{s(n)}g_{s(n)}^m\right) \cup \left(\varepsilon_j g_j^m\right) \cup (0)$$

and observe that, by (1) and (5),

(6) $$C_0(K_m) = \left(\bigoplus_{i=0}^{m-1} c_0^i\right) \oplus C_0(K^m),$$

where $c_0^i = c_0$.

Since F_m is the completion of $c_0 = \left(\bigoplus_{i=0}^{\infty} c_0^i\right)_{c_0}$ for the norm $|\cdot|_m$ and since l_i^1 is w^*-closed in l^1 for all $i \geq 0$, we see from (2) that we may write

$$F_m = \left(\bigoplus_{i=0}^{m-1} c_0^i\right) \oplus G_m, \tag{7}$$

where G_m is the completion of $\left(\bigoplus_{i=m}^{\infty} c_0^i\right)_{c_0}$ for the norm $|\cdot|_m$. Thus, by (6) and (7), it suffices to show that $T_m(G_m)$ is dense in $C_0(K^m)$. For a given $\varphi \in C_0(K^m)$ define the sequence $x = (x_{in})$ by

$$x_{mn} = \varphi\left(f_n^m + \varepsilon_{s(n)} g_{s(n)}^m\right) - \varphi\left(\varepsilon_{s(n)} g_{s(n)}^m\right)$$

and

$$x_{in} = 0 \qquad \text{for } i \neq m.$$

Clearly $x \in c_0^m$ and $\left\langle x, f_n^m + \varepsilon_{s(n)} g_{s(n)}^m \right\rangle = \langle x, f_n^m \rangle = x_{mn}$ for all $n \geq 0$. Further, for each $k \geq 0$ let the sequence $y^k = \left(y_{ij}^k\right) \in c_0^{m+1}$ be given by

$$y_{ij}^k = 0\,(i \neq m+1), \quad y_{m+1,j}^k = \frac{1}{\varepsilon_j} \varphi\left(\varepsilon_j g_j^m\right)\ (j \leq k), \quad y_{m+1,j}^k = 0\,(j > k),$$

so that for $s(n) = j$

$$\begin{aligned}\left\langle y^k, f_n^m + \varepsilon_{s(n)} g_{s(n)}^m \right\rangle &= \left\langle y^k, \varepsilon_j g_j^m \right\rangle = \varepsilon_j \left\langle y^k, f_j^{m+1} \right\rangle \\ &= \varepsilon_j y_{m+1,j}^k = \begin{cases} \varphi(\varepsilon_j g_j^m) & j \leq k \\ 0 & j > k \end{cases}.\end{aligned}$$

Then $x + y^k \in c_0^m \oplus c_0^{m+1} \subset G_m$ for all $k \geq 0$ and it is easy to check that

$$T_m\left(x + y^k\right) = \left(x + y^k\right)_{|K_m} = \left(x + y^k\right)_{|K^m} \longrightarrow \varphi$$

as $k \longrightarrow \infty$. We conclude that $T_m(G_m)$ is dense in $C_0(K^m)$, which completes the proof.

Remark 1.2. Recall the natural inclusions $i_m \colon F_{m+1} \longrightarrow F_m$; then $F = \operatorname{proj}_m (F_m, i_m)$ However, Theorem 1.1 *does not* say that F is the projective limit of the sequence $(C_0(K_m))$ with respect to inclusion maps, since there are not such maps between the spaces $C_0(K_m)$. In order to obtain such a representation, we proceed as follows. Recalling (1), write K_{m+1} as

$$K_{m+1} = (f_n^0) \cup \ldots (f_n^m) \cup (g_n^m) \cup \left(\varepsilon_j g_j^{m+1}\right) \cup (0)$$

and introduce the *linearization maps* $j_m: C_0(K_{m+1}) \longrightarrow C_0(K_m)$ defined as follows: if $\varphi \in C_0(K_{m+1})$ then, for all $n \geq 0$,

$$\begin{aligned} &j_m\varphi\left(f_n^i\right) = \varphi\left(f_n^i\right) \qquad \text{for } i = 0, 1, \ldots, m-1, \\ &j_m\varphi\left(f_n^m + \varepsilon_{s(n)} g_{s(n)}^m\right) = \varphi\left(f_n^m\right) + \varepsilon_{s(n)}\varphi\left(g_{s(n)}^m\right), \\ &j_m\varphi\left(\varepsilon_j g_j^m\right) = \varepsilon_j\varphi\left(g_j^m\right). \end{aligned} \tag{8}$$

Then we have

Proposition 1.3. $F = \mathrm{proj}_m\left(C_0\left(K_m\right), j_m\right)$.

Proof. Considering the diagram

$$\begin{array}{ccc} F_{m+1} & \xrightarrow{i_m} & F_m \\ T_{m+1} \downarrow & & \downarrow T_m \\ C_0(K_{m+1}) & \xrightarrow[j_m]{} & C_0(K_m) \end{array}$$

we have, for $x \in F_{m+1}$,

$$j_{m+1}T_{m+1}x = j_m\left(x_{|K_{m+1}}\right) = x_{|K_m} = T_m i_m x,$$

since x is linear on K_m. Therefore, the family (T_m) defines an isomorphism T of F onto $\mathrm{proj}_m\left(C_0\left(K_m\right), j_m\right)$.

Next we observe the following

Proposition 1.4. $C_0\left(K_m\right) \simeq c_0$ for all $m \geq 0$.

Proof. Recalling (6) we see that it suffices to show the isomorphism $C_0\left(K^m\right) \simeq c_0$. But K^m is homeomorphic to

$$K_0 = \left(f_n^0 + \varepsilon_{s(n)} g_{s(n)}^0\right) \cup \left(\varepsilon_j g_j^0\right) \cup (0)$$

and hence it is enough to prove that $C_0\left(K_0\right) \simeq c_0$. Define a map $S_0: C_0\left(K_0\right) \longrightarrow c_0 \oplus c_0$ as follows:

$$S_0\varphi = (x, y), \qquad \varphi \in C_0\left(K_0\right),$$

where

$$x = \left(\varphi\left(f_n^0 + \varepsilon_{s(n)} g_{s(n)}^0\right) - \varphi\left(\varepsilon_{s(n)} g_{s(n)}^0\right)\right), \qquad y = \left(\varphi\left(\varepsilon_n g_n^0\right)\right).$$

It is immediate to see that S_0 is a linear, continuous and one–to–one. Moreover, if $(x, y) \in c_0 \oplus c_0$ define φ on K_0 by

$$\varphi\left(f_n^0 + \varepsilon_{s(n)} g_{s(n)}^0\right) = x_n + y_{s(n)}, \quad \varphi\left(\varepsilon_j g_j^0\right) = y_j, \qquad \varphi(0) = 0.$$

Then clearly $\varphi \in C_0(K_0)$ and $S_0\varphi = (x, y)$, so that S_0 is also onto.

Remark 1.5. Another representation of $C_0(K_0)$ may be obtained as follows. For all $j \geq 0$ put $N_j = \{n \geq 0 : s(n) = j\}$ and let αN_j be the one–point compactification of N_j. Then it is immediate to check that $C_0(K_0) = \left(\bigoplus_j C(\alpha N_j)\right)_{c_0} \equiv \left(\bigoplus_j c\right)_{c_0}$.

We conclude this section with the following

Proposition 1.6. *If X is an infinite-dimensional Banach subspace of F, then $X \supset c_0$. If X is complemented in F, then $X \simeq c_0$.*

Proof. If X is Banach, then it is a subspace of some F_m and the first assertion follows from Theorem 1.1 and Proposition 1.4. For the second assertion, let $P: F \longrightarrow X$ be a continuous projection and let m be such that $\|Px\| \leq c|x|_m$ for all $x \in F$ ($\|\cdot\|$ is the norm of X). Then the latter inequality holds also for all $x \in F_m$ by continuity, so that X is complemented in F_m. Now it suffices to apply again Theorem 1.1 and Proposition 1.4.

2. F is a space of continuous functions.

Now we consider the compact sets K_m defined in (1) and we put $H_m = \bigcup_{i=0}^{m} K_i$, so that $K = \bigcup_m H_m = \bigcup_m K_m$. It is not difficult to see that K is compact for the w^*–topology induced by l^1, which is not good for our purposes, of course. Therefore, we are going to put on K a stronger topology. Precisely, each set H_m is compact in the induced w^*–topology and we take on K the finest topology τ making each canonical inclusion $H_m \longrightarrow K$ continuous, i.e. the inductive topology with respect to the family $(H_m, H_m \longrightarrow K)$ (cf. [1]). It is known, but also not difficult to verify directly, that, for a space (K,τ) which is the inductive limit of an increasing sequence (H_m) of compact subsets, we have:

(i) each H_m is compact in (K,τ) and each compact subset of (K,τ) is contained in some H_m;

(ii) (K,τ) is normal, hence completely regular;

(iii) (K,τ) is a k-space, i.e. a subset $A \subset K$ is open if and only if A intersects every compact subset of (K,τ) in a relatively open set;

(iv) a function f on K is τ-continuous if and only if f is continuous on each compact subset of (K,τ);

(v) the space $C(K,\tau)$ is complete for the topology of compact convergence.

Remark 2.1. It is clear that, because of (i), it is enough to test (iii)$\div$ (v) against all compact sets H_m. In particular, it follows that $C(K,\tau)$ is a quojection.

In our particular case, the space (K,τ) has, of course, properties (i) $\div$ (v) above, but it is not locally compact, as it can be verified directly (the point 0 does not have a compact neighbourhood). However, we still have

Proposition 2.2. $C(K,\tau) \simeq (c_0)^{\mathbf{N}}$.

Proof. $C(K,\tau)$ is the projective limit of the spaces $C(H_m)$ with respect to the restriction maps $R_m\colon C(H_{m+1}) \longrightarrow C(H_m)$.

We have $H_{m+1} \sim H_m = (f_n^m) \cup (g_n^m) \cup (\varepsilon_j g_j^m)$, hence, if we put $L_m = (H_{m+1} \sim H_m) \cup (0)$, then $\ker R_m$ can be identified with $C_0(L_m)$, i.e. the space of continuous functions on the compact set L_m vanishing at 0. Since L_m is homeomorphic to K_1 for all m, it follows from Proposition 1.4 that $\ker R_m \simeq c_0$, hence $\ker R_m$ is complemented in $C(K,\tau)$ for all m and the result follows.

We shall now define the subspace $C_{a,0}(K,\tau)$ of $C(K,\tau)$ of all affine functions vanishing at $0 \in K$. Precisely, $\varphi \in C_{a,0}(K,\tau)$ if and only if the following holds:

$$(9)\qquad \begin{array}{ll} \varphi \in C(K,\tau) & \text{and } \varphi(0) = 0, \\ \text{if } x, y, x+y \in K, & \text{then } \varphi(x+y) = \varphi(x) + \varphi(y), \\ \text{if } x \text{ and } \lambda x \in K & \text{for some scalar } \lambda, \text{ then } \varphi(\lambda x) = \lambda\varphi(x). \end{array}$$

Then we are in the position to state and prove our main result.

Theorem 2.3. $F \simeq C_{a,0}(K,\tau)$.

Proof. By Proposition 1.3 we may identify F with a space of functions φ on K such that, if $\varphi_m = \varphi_{|K_m}$, then $j_m \varphi_{m+1} = \varphi_m$. Since each φ_m is continuous on K_m, $\varphi \in C(K,\tau)$ by (iv). Moreover, φ satisfies (9) by (8), hence $\varphi \in C_{a,0}(K,\tau)$. Conversely, if $\varphi \in C_{a,0}(K,\tau)$ then $\varphi_m = \varphi_{|K_m} \in C_0(K_m)$ and by (9) $j_m\varphi_{m+1} = \varphi_m$ for all $m \geq 0$. Thus $\varphi \in F$ by Proposition 1.3.

Corollary 2.4. *The strong dual F' is complemented in $C(K,\tau)'$.*

Proof. By Theorem 4 of [5] $F' \simeq \bigoplus_n l^1$. But F' is also a quotient of the strict (LB)-space $C(K,\tau)'$, hence F' is complemented in $C(K,\tau)'$ by [3, Corollary 3.4].

Remark 2.5. Put for all $m \geq 0$

$$J_m = j_{m|c_0^m \oplus C_0(K^{m+1})},$$

then by (8) we have

$$J_m(x+y) = P_m x + J_m(x+y) \qquad \text{for } x \in \textstyle\bigoplus_{i=0}^m c_0^i,\ y \in C_0(K^{m+1}),$$

where P_m is the canonical projection of $\bigoplus_{i=0}^{m} c_0^i$ onto $\bigoplus_{i=0}^{m-1} c_0^i$.

Now, defining the maps $S_m : C_0(K^m) \longrightarrow c_0^m$ as in Proposition 1.4 and considering the diagram

$$\begin{array}{ccc} \bigoplus_{i=0}^{m+1} c_0^i & \xrightarrow{P_m+\overline{S}_m} & \bigoplus_{i=0}^{m} c_0^i \\ {\scriptstyle I_m+S_{m+1}^{-1}}\downarrow & & \downarrow{\scriptstyle I_{m-1}+S_m^{-1}} \\ \left(\bigoplus_{i=0}^{m} c_0^i\right) \oplus C_0(K^{m+1}) & \xrightarrow[j_m]{} & \left(\bigoplus_{i=0}^{m-1} c_0^i\right) \oplus C_0(K^m) \end{array}$$

where I_m is the identity map of $\bigoplus_{i=0}^m c_0^i$ and $\overline{S}_m$ is given by

$$\overline{S}_m(x+y) = S_m^{-1} J_m\left(x + S_{m+1}^{-1} y\right) \qquad \text{for } x \in \textstyle\bigoplus_{i=0}^m c_0^i,\ y \in c_0^{m+1},$$

we have for all $m \geq 0$

$$j_m\left(I_m + S_{m+1}^{-1}\right) = \left(I_{m-1} + S_m^{-1}\right)\left(P_m + \overline{S}_m\right).$$

Therefore, the family $\left(I_{m-1} + S_m^{-1}\right)$ defines an isomorphism S of F onto $\mathrm{proj}_m(\prod_{i=1}^m c_0^i, P_m + \overline{S}_m)$. This shows that our prequojection F can be directly constructed by the method employed in [4, Theorem 1].

REFERENCES

[1] Bourbaki N., *Eléments de Mathématique. Livre III. Topologie générale*, Hermann, Paris, (1958–1961).

[2] Dierolf S. and V. B. Moscatelli, *A note on quojections*, Funct. Approx. Comment. Math., **17**, 131–138, (1987).

[3] Metafune G. and V. B. Moscatelli, *Complemented subspaces of sums and products of Banach spaces*, Ann. Math. Pura Appl. **153**, 175–190, (1988).

[4] Metafune G. and V. B. Moscatelli, *Prequojections and their duals*, Progress in Functional Analysis, (eds.: K. D. Bierstedt and J. Bonet), North-Holland Math. Stud., (1991).

[5] Moscatelli V. B., *Strongly non-norming subspaces and prequojections*, Studia Math. 95, 249–254, (1990).

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