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A NOTE ON NONDISTINGUISHED KOTHE SPACES OF INFINITE TYPE

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ABSTRACT: In this note it is proved that if $\lambda_\infty(A)$ is any nondistinguished Köthe echelon space of infinite order, then there is a linear form on its strong dual $(\lambda_\infty(A))'_b$ which is locally bounded (i.e. bounded on the bounded sets) but not continuous.

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A Fréchet space F is distinguished if its strong dual F'_b is barrelled or, equivalently, bornological. This means that the canonical representation of F as the (reduced) projective limit $\text{proj}_{\leftarrow n} F_n$ leads to the representation $\text{ind}_{n \rightarrow} F'_n$ of F'_b as the inductive limit of the dual spectrum $(F')_n$. Distinguished Fréchet spaces were introduced by Dieudonné, Schwartz and Grothendieck.

The first example of a nondistinguished Fréchet space was given by Köthe

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and Grothendieck and it was the Köthe echelon space of order one $\lambda_1(A)$ for the Köthe matrix $A=(a_n)_{n\in\mathbb{N}}$ on the index set $\mathbb{N}\times\mathbb{N}$ given by $a_n(k,j)=j$ if $k\leq n$ and $a_n(k,j)=1$ if $k\geq n+1$. For this matrix Grothendieck even proved that there is a linear form on $(\lambda_1(A))'_b$ which is locally bounded but not continuous. Distinguished echelon spaces of order one were characterized in terms of the Köthe matrix A in [3]. In [2] it is shown that all nondistinguished Köthe echelon spaces of order one share the bad behaviour of the Köthe-Grothendieck example. On the other hand, an example of Komura (see e. g. [6] p.292) shows that there are nondistinguished Fréchet spaces such that every locally bounded linear form on F'_b is continuous. More examples of both types of nondistinguished Fréchet spaces can be seen in [5], where one can also find the following characterization of the quasibarrelled spaces E such that on the strong dual E'_b, there exists a noncontinuous locally bounded linear form

Lemma ([5]): *Let* E *be a quasibarrelled l.c.s., Then the following statements are equivalent:*

(i) there exists a noncontinuous locally bounded linear form on E'_b

(ii) there is a filter $\mathcal{F}$ *in* E *such that*

(a) for every 0*-neighbourhood* U *there is* $\rho_U>0$ *with* $\rho_U U\in\mathcal{F}$

(b) for every bounded set B *there exists a closed* 0*-neighbourhood in* $(E,\sigma(E,E'))$, V_B, *such that* $E\setminus(B+V_B)\in\mathcal{F}$

On the other hand, distinguished Köthe echelon spaces $\lambda_\infty(A)$ were characterized in terms of the Köthe matrix A in [1]. Making use of these two results we show that all nondistinguished echelon spaces $\lambda_\infty(A)$ present exactly the same behaviour as in the case of echelon spaces of order one.

Notations for Köthe spaces are as in [4]. We recall that a Köthe space $\lambda_\infty(A)$ has a fundamental system of bounded sets of the form

$$\bar{v}(l_\infty)_I:=\{\, z\in K^I:\ z=\bar{v}z';\ \sup_{i\in I}|z'_i|\leq 1\,\},$$

where $\bar{v}=\inf_m \rho_m a_m^{-1}$ for some sequence of positive numbers ρ_m.

Theorem: *Let* $\lambda_\infty(A)$ *be a nondistinguished Köthe echelon space of infinite order. Then there is a locally bounded noncontinuous linear form on* $(\lambda_\infty(A))'_b$.

Proof: First, we assume that A is a Köthe matrix on $\mathbb{N}\times\mathbb{N}$ satisfying

(1) $a_n(k,j)=1 \quad \forall k\geq n,\ \forall j\in\mathbb{N}$

(2) $\lim_{j\to\infty} \frac{1}{a_{n+1}(n,j)} = 0 \quad \forall n\in\mathbb{N}.$

Let $\mathcal{B}_h$ be the family of all the subsets B of $\mathbb{N}\times\mathbb{N}$ of the form $B = \bigcup_{k\geq p} \{k\}\times B_k$, where $p=p(B)\geq h$, and $B_k:=\{1,2,...,n_k\}$, $n_k\in\mathbb{N}$.

Now, given $h\in\mathbb{N}$ and $(j_k)_{k\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$ we put

$$M(h,(j_k)_{k\geq h}):=\{\ \chi_B:\ B\in\mathcal{B}_h \text{ and } j_k\in B_k \text{ for } k\geq p(B)\},$$

where χ_B denotes the characteristic function of B.

Hence $\{\ M(h,(j_k)_{k\geq h}):h\in\mathbb{N},\ (j_k)_{k\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}\}$ generates a filter $\mathcal{F}$ in $\lambda_\infty(A)$. We will see that $\mathcal{F}$ satisfies the conditions (a) and (b) in Lemma above.

Given $n\in\mathbb{N}$, since $a_n(k,j)=1$ for all $k\geq n$, we have that

$$M(h,(j_k)_{k\geq h})\subseteq U_n:=\{(x_{kj}):\ \sup_{k,j}\ a_n(k,j)\,|x_{kj}|\leq 1\}$$

for all $h\geq n$ and arbitrary $(j_k)_{k\geq h}$. Therefore, condition (a) holds.

To check the other condition, we take a bounded set in $\lambda_\infty(A)$. We may assume that it is of the form $\bar{v}(l_\infty)_1$. Using (2), we obtain that

$$\forall k\in\mathbb{N}\ \exists j_k\in\mathbb{N}\ \forall j\geq j_k :\ |x_{kj}|< 1/4 \quad \forall x=(x_{kj})\in\bar{v}(l_\infty)_1,$$

On the other hand, if x belongs to the convex hull of $M(1,(j_k)_{k\in\mathbb{N}})$, x can be expressed as a convex sum $x = \kappa_1\chi_{B_1}+...+\kappa_m\chi_{B_m}$, where $\chi_{B_r}\in M(1,(j_k)_{k\in\mathbb{N}})$, $\kappa_r>0$, $1\leq r\leq m$, and $\kappa_1+...+\kappa_m=1$. By the definition of $M(1,(j_k)_{k\in\mathbb{N}})$, we may find $h\geq 1$ such that $j_p\in B_{r,p}$ for all $p\geq h$, and $1\leq r\leq m$, whence, $x(p,j_p)=1$ for all $p\geq h$. Therefore,

$$(2\bar{v}(l_\infty)_1+\frac{1}{4}\ U_1(<1))\cap(co(M(1,(j_k)_{k\in\mathbb{N}})) + \frac{1}{4}\ U_1(<1))=\emptyset$$

where $U_1(<1)$ denotes the open unit ball for a_1. Since these are disjoint open sets, of which the first one is absolutely convex and the second one convex, we can find $u \in (\lambda(A))'$ such that $u \in (2\tilde{v}(l_\infty)_1 + \frac{1}{4}U_1(<1))^\circ$ and $|u(x)|>1$ for each $x \in co(M(1,(j_k)_{k\in\mathbb{N}})) + \frac{1}{4}U_1(<1))$. Therefore,

$$M(1,(j_k)_{k\in\mathbb{N}}) \cap (\tilde{v}(l_\infty)_1 + \frac{1}{2}\{u\}^\circ) = \emptyset,$$

Now we apply the former Lemma to obtain that there is a noncontinuous locally bounded linear form on $(\lambda_\infty(A))'_b$

In the general case, if $\lambda_\infty(A)$ is not distinguished, according with [1] and [2], it has a sectional subspace isomorphic to $\lambda_\infty(B)$ where B is a Köthe matrix on $\mathbb{N}\times\mathbb{N}$ satisfying (1) and (2) above. Since sectional subspaces are complemented, we are done. ∎

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