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THE CLASS OF DOUBLE MS-ALGEBRAS SATISFYING THE COMPLEMENT PROPERTY

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Abstract

We construct a functor from the category of de Morgan algebras into the category of double MS-algebras which satisfy a certain condition, called the complement property and show that this functor has a left adjoint.

1. Introduction

An *MS-algebra* $<A, \vee, \wedge, {}^0, 0, 1>$ is an algebra of type $<2,2,1,0,0>$ such that $<A, \vee, \wedge, 0, 1>$ is a bounded distributive lattice and ^{0} is a unary operation satisfying $x \leqslant x^{00}$, $(x \wedge y)^0 = x^0 \vee y^0$, $1^0 = 0$. These algebras belong to the class of Ockham algebras introduced by Berman [1]. A *double MS-algebra* is an algebra $< A, {}^0, {}^+>$ such that $<A, {}^0>$ and its dual $<A_d, {}^+>$ are MS-algebras and for every $x \in L$, $x^{0+} = x^{00}$, $x^{+0} = x^{++}$. T. S. Blyth and J. C. Varlet in [3] pointed out that every de Morgan algebra L can be represented non-trivially as the skeleton of the double MS-algebra $L^{[2]} = \{(a, b) \in L \times L : a \leqslant b\}$. In this paper we consider the class of double MS-algebras satisfying the complement property which is related to the construction due to T. S. Blyth and J. C. Varlet and investigate the role of the complete-

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ment property in the theory of double MS-algebras. Our main result is that the complement property associates a double MS-algebra **K** (L) with each de Morgan algebra L. More precisely, we show that **K** is a functor from the category **M** of de Morgan algebras and de Morgan algebra homomorphisms into the category **V** of double MS-algebras which satisfy a certain condition called the complement property and double MS—algebra homomorphisms and that the functor **K** has a left adjoint **L**. As a consequence of the construction we obtain an important property of the left adjoint **L** of **K**: for each double MS-algebra A satisfying the complement property, the congruence lattice of A is isomorphic to the congruence lattice of the lattice $\mathbf{L}(A)$.

The main tools we use in the proof of the results mentioned above are the duality between double MS-algebras and certain ordered topological spaces developed by T. S. Blyth and J. C. Varlet in [5]. We recall here the main results that we shall need.

Since double MS-algebras are bounded distributive lattices, they are dually equivalent to some suitable category of Priestley spaces (i. e., compact totally order disconnected spaces) and order-preserving continuous functions. In fact, a *double MS-space* is a Priestley space X endowed with two continuous order-reversing maps $g_1, g_2 : X \to X$ satisfying the following conditions:

(1) $g_1^2(x) \leqslant x$

(2) $g_2^2(x) \geqslant x$

(3) $g_2(g_1(x)) = g_2^2(x)$

(4) $g_1(g_2(x)) = g_1^2(x)$.

If $< X, g_1, g_2 >$ is a double MS-space, then we can define two unary operations ^{0} , $^+$ on $O(X)$, the lattice of clopen decreasing sets of X , by setting

$$I^0 = X \backslash g_1^{-1}(I),\ I^+ = X \backslash g_2^{-1}(I)$$

for each $I \in O(X)$, and thereby obtain a double MS-algebra. Conversely, if $< A, ^0, ^+ >$ is a double MS-algebra, then we can define two maps g_1, g_2 on the ordered set $X(A)$ of prime ideals of A by setting

$$g_1(p) = \{a \in A : a^0 \notin p\},\ g_2(p) = \{a \in A : a^+ \notin p\}$$

for each $p \in X(A)$, and thereby obtain a double MS-space, which will be denoted by $\mathbf{S}(A)$. These constructions give a dual equivalence.

A subset Y of a double MS-space $< X, g_1, g_2 >$ is said to be a g_I - *invariant subset* if it satisfies the condition

$$x \in Y \Rightarrow g_i(x) \in Y,\ i \in I = \{1,2\}.$$

According to Urquhart in [8], if A is a double MS-algebra, then the congruence lattice of the MS—algebra $< A, ^0 >$ is dually isomorphic to the lattice of all closed g_1 - invariant subsets of the MS-space $< X, g_1 >$. Duallly, the congruence lattice of the dual MS-algebra $< A, ^+ >$ is dually isomorphic to the lattice of all closed g_2 - invariant subsets of the dual MS-space $< X, g_2 >$. Therefore, the congruence lattice of the double MS-algebra A is dually isomorphic to the lattice of all closed g_I -invariant subsets of the double MS-space $< X, g_1, g_2 >$. If $\theta(Y)$ is the congruence associated with the closed g_I -invariant subset Y , then

$$b \equiv c(\theta(Y)) \text{ iff } B \cap Y = C \cap Y$$

where B, C are the clopen decreasing subsets that represent b, c.

Clearly, $g_1(X)$ is a closed g_1-invariant subset of X and $g_1(X) = g_1^2(X)$, $g_2(X)$ is a closed g_2-invariant subset of X and $g_2(X) = g_2^2(X)$.

2. The complement property

Definition 2.1. A double MS-space $< X, g_1, g_2 >$ is said to satisfy the *complement property* if the equations $g_1(X) \cup g_2(X) = X$, $g_1(X) \cap g_2(X) = \varnothing$ hold. In other words, X is the disjoint union of $g_1(X)$ and $g_2(X)$. A double MS-algebra A is said to satisfy the complement property if the double MS-space $S(A)$ satisfies it.

In what follows the complement property will be denoted by (CP).

Example 2.1. For every Boolean algebra $< B, \vee, \wedge, ', 0, 1 >$, let $B^{[2]} = \{(a, b) \in L \times L : a \leqslant b\}$. $B^{[2]}$ is a double Stone algebra which satisfies (CP), where the pseudocomplement of (a,b) is (b', b'), the dual pseudocomplement of (a,b) is (a', a').

Theorem 2.1. *Let A be a double MS-algebra and $< X, g_1, g_2 > = S(A)$. Then the following are equivalent conditions:*

(i) A satisfies (CP).

(ii) Given b, c in A such that $b = b^{00}$, $c = c^{00}$ and $b \leqslant c$, then there exists a unique element $d \in A$ such that $d^{++} = b$, $d^{00} = c$.

Proof. We are going to show that the following conditions (1) and (3) are equivalent, and that so are conditions (2) and (4).

(1) $g_1(X) \cup g_2(X) = X$

(2) $g_1(X) \cap g_2(X) = \varnothing$

(3) Given b, c in A such that $b^{00} = c^{00}$, $b^{++} = c^{++}$, then $b = c$.

(4) Given b, c in A such that $b = b^{00}$, $c = c^{00}$, $b \leqslant c$, then there exists an element $d \in A$ such that $d^{++} = b$, $d^{00} = c$.

The following B, C, D represent b, c, d respectively.

(1)$\Leftrightarrow$(3). Since $b^{++} = c^{++}$, $b^{00} = c^{00}$ iff $b^{+} = c^{+}$, $b^{0} = c^{0}$

iff $X \backslash g_2^{-1}(B) = X \backslash g_2^{-1}(C)$, $X \backslash g_1^{-1}(B) = X \backslash g_1^{-1}(C)$

iff $g_2^{-1}(B) = g_2^{-1}(C)$, $g_1^{-1}(B) = g_1^{-1}(C)$

iff $g_2^{-1}[(B \backslash C) \cup (C \backslash B)] = \varnothing$, $g_1^{-1}[(B \backslash C) \cup (C \backslash B)] = \varnothing$

iff $(B \backslash C) \cup (C \backslash B) \subseteq X \backslash g_2(X)$, $(B \backslash C) \cup (C \backslash B) \subseteq X \backslash g_1(X)$

iff $(B \backslash C) \cup (C \backslash B) \subseteq X \backslash (g_1(X) \cup g_2(X))$

iff $B \cap (g_1(X) \cup g_2(X)) = C \cap (g_1(X) \cup g_2(X))$

iff $b \equiv c \; (\theta(g_1(X) \cup g_2(X)))$

If (3) holds, that is, $c = b$, then $\theta(g_1(X) \cup g_2(X)) = \omega$ (i. e., the zero congruence), $g_1(X) \cup g_2(X) = X$. If (1) holds, that is, $g_1(X) \cup g_2(X) = X$, then $\theta(g_1(X) \cup g_2(X)) = \omega$, $c \equiv b(\theta(g_1(X) \cup g_2(X))$ implies $c = b$.

(2)$\Leftrightarrow$(4). If (2) holds and $b = b^{00}$, $c = c^{00}$, $b \leqslant c$, then $B = X \backslash g_1^{-1}(X \backslash g_1^{-1}(B))$, $C = X \backslash g_1^{-1}(X \backslash g_1^{-1}(C))$, $B \subseteq C$, that is, $B = g_1^{-2}(B)$, $C = g_1^{-2}(C)$. Setting

$D = (C \cap g_1(X)) \cup B$, therefore we have that D is a clopen decreasing set of $S(A)$. $g_1(X) \cap g_2(X) = \varnothing$ implies that $D \backslash B \subseteq C \cap g_1(X) \subseteq g_1(X) \subseteq X \backslash g_2(X)$, $C \backslash D = C \backslash (B \cup g_1(X)) \subseteq C \backslash g_1(X) \subseteq X \backslash g_1(X)$, hence we obtain that $X \backslash g_2^{-1}(D) = X \backslash g_2^{-1}(B)$, $X \backslash g_1^{-1}(C) = X \backslash g_1^{-1}(D)$, that is, $d^+ = b^+$, $d^0 = c^0$, $d^{++} = b^{++} = b$, $d^{00} = b^{00} = c$. Therefore (4) holds.

If (4) holds and there exists an element $d \in A$ such that $d^{++} = b = b^{00}$, $d^{00} = c = c^{00}$, that is, $d^+ = b^+$, $d^0 = c^0$, then $X \backslash g_2^{-1}(D) = X \backslash g_2^{-1}(B)$, $X \backslash g_1^{-1}(D) = X \backslash g_1^{-1}(C)$, hence, we have that $(B \backslash D) \cup (D \backslash B) \subseteq X \backslash g_2(X)$, $(C \backslash D) \cup (D \backslash C) \subseteq X \backslash g_1(X)$. It follows that $B \subseteq D \subseteq C$ from the fact that $b = d^{++} \leqslant d \leqslant d^{00} = c$, and so $D \backslash B \subseteq X \backslash g_2(X)$, $C \backslash D \subseteq X \backslash g_1(X)$, $C \backslash B \subseteq X \backslash (g_1(X) \cap g_2(X))$. Fix $b = 0$, $c = 1$, then $B = \varnothing$, $C = X$, it follows that $g_1(X) \cap g_2(X) = \varnothing$. □

Theorem 2.2. *Let $\langle L, \vee, \wedge, ', 0, 1 \rangle$ be a de Morgan algebra and let $\mathbf{K}(L) = \{(a, b) \in L \times L : a \leqslant b\}$. For every $(a,b) \in \mathbf{K}(L)$, define $(a,b)^0 = (b',b')$ and $(a, b)^+ = (a',a')$. Then $\langle \mathbf{K}(L), ^0, ^+ \rangle$ is a double MS-algebra satisfying (CP).*

Proof. By Theorem 2.3 in [3], $\langle \mathbf{K}(L), ^0, ^+ \rangle$ is a double MS-algebra.

Let (a,b), $(c,d) \in \mathbf{K}(L)$ such that $(a,b) = (a,b)^{00}$, $(c,d) = (c,d)^{00}$ and $(a,b) \leqslant (c,d)$. Then $a = b$, $c = d$ and $a \leqslant c$, hence $(a,c) \in \mathbf{K}(L)$, $(a,c)^{++} = (a,a)$, $(a, c)^{00} = (c,c)$. Suppose that there exists some $(a_1,c_1) \in \mathbf{K}(L)$ such that $(a_1,c_1)^{++} = (a,c)^{++} = (a,a)$, $(a_1,c_1)^{00} = (a,c)^{00} = (c,c)$. This means that $a = a_1$, $c = c_1$, i. e., $(a,c) = (a_1,c_1)$. According to Theorem 2.1, $\mathbf{K}(L)$ satisfies (CP). □

Lemma 2.1. *Let L, L_1 be de Morgan algebras and let $h : L \to L_1$ be a 0-1-preserving de Morgan algebra homomorphism. For each (x,y) in $\mathbf{K}(L)$ define $\mathbf{K}(h)((x,y)) = (h(x),h(y))$. Then $\mathbf{K}(h)$ is a double MS-algebra homomorphism from $\mathbf{K}(L)$ into $\mathbf{K}(L_1)$.* □

According to Theorem 2.2 and Lemma 2.1, we obtain a functor $\mathbf{K}$ from the category of de Morgan algebras and de Morgan algebra homomorphisms into the category of double MS-algebras and MS-algebra homomorphisms.

Lemma 2.2. *Let A be a double MS-algebra and let $\mathbf{L}(A) = \{x \in A : x = x^{00}\}$. Then $\mathbf{L}(A)$ is a de Morgan algebra.* □

Lemma 2.3. *Let A, A_1 be double MS-algebras and let h be a double MS-algebra homomorphism, define $\mathbf{L}(h)(x) = h(x)$ for each $x \in \mathbf{L}(A)$. Then $\mathbf{L}(h)$ is a de Morgan algebra homomorphism from $\mathbf{L}(A)$ into $\mathbf{L}(A_1)$.*

Proof. It suffices to show that the definition of $\mathbf{L}(h)$ is reasonable. For $x \in \mathbf{L}(A)$, $x \in A$ and $x = x^{00}$, We have $h(x) = h(x^{00}) = h(x)^{00}$, thus $h(x) \in \mathbf{L}(A_1)$. □

MS-algebras and double MS-algebra homomorphisms into the category of de Morgan algebras and de Morgan algebra homomorphisms.

The following will be devoted to prove that **L** is a left adjoint of **K**.

Theorem 2.3. *Let L be a de Morgan algebra and let* $\mathbf{L}(\mathbf{K}(L)) = \{x \in K(L) : x = x^{00}\}$. *Then* $\mathbf{L}(\mathbf{K}(L)) \cong L$.

Proof. It is plain that $\mathbf{L}(\mathbf{K}(L))$ is a de Morgan algebra, whose elements have the form (a,a), $a \in L$. Define $P_L : (a,a) \to a$. It is easy to show that P_L is an isomorphism from $\mathbf{L}(\mathbf{K}(L))$ into L. □

Theorem 2.4. *Let A be a double MS-algebra satisfying (CP). Define* $J_A(a) = (a^{++}, a^{00})$ *for each $a \in A$. Then J_A is a double MS-algebra isomomorphism from A into* $\mathbf{K}(\mathbf{L}(A))$.

Proof. Since $(a^{++})^{00} = a^{++}$ and $(a^{00})^{00} = a^{00}$, we deduce that $a^{++}, a^{00} \in \mathbf{L}(A)$. By observing that $a^{++} \leqslant a^{00}$, we have $(a^{++}, a^{00}) \in \mathbf{K}(\mathbf{L}(A))$, which implies that the definition of J_A is reasonable. Since

$$J_A(a \vee b) = ((a \vee b)^{++}, (a \vee b)^{00}) = (a^{++} \vee b^{++}, a^{00} \vee b^{00})$$
$$= (a^{++}, a^{00}) \vee (b^{++}, b^{00}) = J_A(a) \vee J_A(b),$$
$$J_A(a \wedge b) = ((a \wedge b)^{++}, (a \wedge b)^{00}) = (a^{++} \wedge b^{++}, a^{00} \wedge b^{00})$$
$$= (a^{++}, a^{00}) \wedge (b^{++}, b^{00}) = J_A(a) \wedge J_A(b),$$

J_A is a lattice homomorphism.

Moreover, $J_A(a^0) = (a^{0++}, a^{000}) = (a^0, a^0)$, $(J_A(a))^0 = (a^{++}, a^{00})^0 = (a^{000}, a^{000}) = (a^0, a^0)$, hence $J_A(a^0) = (J_A(a))^0$. Similarly, $J_A(a^+) = (J_A(a))^+$. So we obtain that J_A is a double MS-algebra homomorphism. To see that J_A is also an isomorphism, suppose $(a^{++}, a^{00}) = (b^{++}, b^{00})$. By Theorem 2.1, we have $a = b$. This means that J_A is one-one. Since, for any $(x,y) \in \mathbf{K}(\mathbf{L}(A))$, that is, $x, y \in \mathbf{L}(A)$ and $x \leqslant y$, by Theorem 2.1, there exists some $c \in A$ such that $c^{++} = x$, $c^{00} = y$, and hence, $J_A(c) = (x,y)$. This means that J_A is an onto mapping. □

According to Theorems 2.3 and 2.4, it is easy to check that the mappings J_A and P_L define natural transformations $J : 1_V \to \mathbf{KL}$ and $P : \mathbf{LK} \to 1_M$, where 1_V and 1_M are the identity functors in the categories **V** and **M** respectively. More precisely, we have:

Theorem 2.5. $<$**L**, **K**, J, $P>$ *is an adjunction, with unit J and counit P.*

Proof. Since we have already noted that $J : 1_V \to \mathbf{KL}$ and $P : \mathbf{KL} \to 1_M$ are natural transformations, according to a result in [7, ch. iv, Theorem 2(v)], to complete the proof, we have to show that the following two conditions hold, where 1_X denotes the identity for the object X:

(1) For each de Morgan algebra L, $\mathbf{K}(P_{\mathbf{LK}(L)})J_{\mathbf{K}(L)} = 1_{\mathbf{K}(L)}$, and

(1) For each de Morgan algebra L, $\mathbf{K}(P_{\mathbf{LK}(L)})J_{\mathbf{K}(L)} = 1_{\mathbf{K}(L)}$, and

(2) For each double MS-algebra A satisfying (CP), $P_{\mathbf{LKL}(A)}(\mathbf{L}(J_A)) = 1_{\mathbf{L}(A)}$.

The prove (1), let a, $b \in L$, $a \leqslant b$. Since

$$J_{\mathbf{K}(L)}(a,b) = ((a,b)^{++}, (a,b)^{00}) = ((a,a), (b,b)),$$

we have

$$\mathbf{K}(P_{\mathbf{LK}(L)})J_{\mathbf{K}(L)}((a,b)) = (P_{\mathbf{LK}(L)}(a,a), P_{\mathbf{LK}(L)}(b,b)) = (a,b).$$

To prove (2), let $a \in \mathbf{L}(A)$, that is, $a \in A$, $a = a^{00}$, since

$$J_A(a) = (a^{++}, a^{00}) = (a,a) \in \mathbf{L}(\mathbf{KL}(A)),$$
$$P_{\mathbf{LKL}(A)}((\mathbf{L}(J_A))(a) = P_{\mathbf{LKL}(A)}((a,a)) = a \qquad \square$$

Theorem 2.6. *Let $< X, g_1, g_2 >$ be a double MS-space satisfying (CP). Then the correspondence $T \to T \cup g_2(T)$ establishes an isomorphism from the lattice of all the closed g_1-invariant subsets of $g_1(X)$ onto the lattice of all the g_I-invariant subsets in X.*

Proof. Since X is a compact totally order-disconnected topological space and $g_1(X)$ is a closed set in X, hence, is compact and the continuous mapping $g_2 |_{g_1(X)}$ is closed. Let T be a closed g_1-invariant subset of $g_1(X)$. It is plain that $T \cup g_2(T)$ is a closed set in X. To prove $T \cup g_2(T)$ is a g_I-invariant subset in X, let $x \in T \cup g_2(T)$. If $x \in T$, then $g_1(x) \in g_1(T) = T$; if $x \in g_2(T)$, let $x = g_2(t)$, $t \in T$, then $g_1(x) = g_1(g_2(t)) = g_1^2(t) \in T$. This implies that $T \cup g_2(T)$ is g_1-invariant. Similarly, $T \cup g_2(T)$ is g_2-invariant, hence, we obtain $T \cup g_2(T)$ is a g_I-invariant subset in X. For each $x \in g_2(T) \cap g_1(X)$, i.e., there exist $t \in T$, $x_0 \in X$ such that $x = g_2(t) = g_1(x_0)$, we have $g_1(x) = g_1(g_2(t)) = g_1^2(t) \in T$, and so, $g_1^2(x) \in T$. Since $g_1^2(x) = g_1^2(g_1(x_0)) = g_1(x_0) = x$, $x \in T$, therefore, $g_2(T) \cap g_1(X) \subseteq T$ and then, $(T \cup g_2(T)) \cap g_1(X) = T$. From this equality we obtain that $T \cup g_2(T) \subseteq S \cup g_2(S)$ iff $T \subseteq S$. Finally, to see that the mapping is onto, note that if Y is a g_I-invariant subset in X, then $Y = (Y \cap g_1(X)) \cup (Y \cap X \backslash g_1(X))$. It is enough to show that the following two conditions hold:

(1) $Y \cap g_1(X)$ is a g_1-invariant subset of $g_1(X)$.

(2) $Y \cap X \backslash g_1(X) = g_2(Y \cap g_1(X))$.

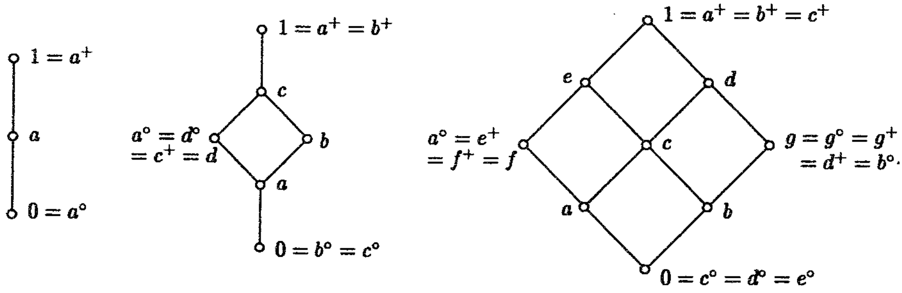
The prove (1), note that if $y \in Y \cap g_1(X)$, then $y \in Y$, $y \in g_1(X)$ and hence $g_1(y) \in Y$, $g_1(y) \in g_1(g_1(X)) \subseteq g_1(X)$, therefore, $g_1(y) \in Y \cap g_1(X)$.

To see (2), note that, by (CP), $g_1(X) \cup g_2(X) = X$ and $g_1(X) \cap g_2(X) = \varnothing$, hence $X \backslash g_1(X) = g_2(g_1(X))$. If $t \in Y \cap X \backslash g_1(X)$, then there exists $x_0 \in X$ such that $t = g_2(g_1(x_0))$. Since Y is g_I-invariant, $g_1(t) \in Y$, and so, $g_1(g_2(g_1(x_0))) = g_1(x_0) \in Y$, $t = g_2(g_1(x_0)) \in g_2(Y \cap g_1(X))$. Suppose alternatively that $t \in g_2(Y \cap g_1(X))$, then there exists $y \in Y$ such that $t = g_2(y)$, $y \in Y \cap g_1(X)$, since Y is g_I-invariant, $t \in Y$, $t = g_2(y) \in g_2(g_1(X))$. i.e., $t \in Y \cap g_2(g_1(X))$. $\square$

From Theorem 2.6 and topological duality for double MS-algebras, we have:

Corollary 2.1. *The lattices Con(A) and Con $(\mathbf{L}(A))$ are isomorphic for each double MS-algebra A satisfying (CP).* $\square$

Remark. Since the only subdirectly irreducible de Morgan algebras are the chains with two and three elements and the four-element de Morgan algebra with two fixed points, from the above Corollary we obtain at once that the only subdirectly irreducible double MS-algebras satisfying (CP) are the following:



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