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ON BORNOLOGICAL $c_0(E)$ SPACES

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SUMMARY

We essentially prove that if E is a locally convex space where every converging sequence is Mackey converging, then $c_0(E)$ is bornological if and only if E is bornological and such that its strong dual E'_β satisfies Pietsch's condition (B).

INTRODUCTION

Throughout this paper E is a Hausdorff locally convex topological vector space (l.c. space, for short) which system of continuous semi-norms is denoted by P. Then

a) E'_β stands for the strong topological dual of E,

b) $c_0(E)$ is the l.c. space obtained by endowing the linear space of the null sequences of E with the system of semi-norms $\{p^{(\infty)} : p \in P\}$ where each $p^{(\infty)}$ is defined by

$$p^{(\infty)}(e) = \sup_m p(e_m), \forall e \in c_0(E).$$

c) $l^1(E)$ is the l.c. space obtained by endowing the linear space of the sequences e_m of E such that $\sum_{m=1}^{\infty} p(e_m) < \infty$ for every $p \in P$ with the system of semi-norms $\{p^{(1)} : p \in P\}$ where each $p^{(1)}$ is defined by

$$p^{(1)}(e) = \sum_{m=1}^{\infty} p(e_m), \forall e \in l^1(E).$$

Let us also recall that if B is an absolutely convex bounded subset of E, E_B is the normed space obtained by endowing the linear hull of B with the gauge $\|.\|_B$ of B. Such a set B is completing, if E_B is a Banach space. A sequence e_m of E is Mackey converging (resp. fast converging) to e_0 in E if there is an absolutely convex bounded (resp. an absolutely convex completing bounded) subset B of E such that $e_m \to e_0$ in E_B.

Then E'_{mc} (resp. E'_{fc}) is the topological dual of E endowed with the system of the semi-norms

$$\sup_m |\langle e_m, e' \rangle|$$

where e_m is a Mackey (resp. fast) converging sequence of E.

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The space E has property (B) of Pietsch (cf. 1.5.5 of [4]) if for every bounded subset B of $l^1(\mathrm{E})$ there is a bounded subset B of E such that

$$\sum_{m=1}^{\infty} \| f_m \|_{\mathrm{B}} < 1, \forall f \in B.$$

Our purpose is to get analogous results to the following ones for $c_0(\mathrm{E})$ to be bornological or ultrabornological.

MARQUINA-SANZ SERNA THEOREM (cf. theorem 2.6 of [2]). *The space $c_0(\mathrm{E})$ is infrabarrelled if and only if* E *is infrabarrelled and such that* E'_β *has property* (B).

Moreover if $c_0(\mathrm{E})$ is infrabarrelled, then one has

$$c_0(\mathrm{E})'_\beta = l^1(\mathrm{E}'_\beta).$$

MENDOZA THEOREM (cf. theorem 6 of [3]). *The space $c_0(\mathrm{E})$ is barrelled if and only if* E *is barrelled and such that* E'_β *has property* (B).

In fact the results of this paper have been obtained independently by the two authors who were trying to get necessary and sufficient conditions. Preprints went around and J. Mendoza made us aware of each other's work. Many persons we know are interested in this question. This is the reason why we proposed this sufficient condition for publication.

THEOREM 1. *Let* E *be a Mackey space where every converging sequence is Mackey converging. Then the following are equivalent* :

(a) $c_0(\mathrm{E})$ *endowed with the Mackey topology is bornological,*

(b) E *is bornological and such that every absolutely summable sequence of* E'_β *is absolutely summable in* E'_B *for some* $\sigma(\mathrm{E}', \mathrm{E})$*-closed absolutely convex and bounded subset* B *of* E'_β.

Proof. (a) $\Rightarrow$ (b). As $c_0(\mathrm{E}) \times \mathrm{E}$ is topologically isomorphic to $c_0(\mathrm{E})$, there is a closed complemented subset F of $c_0(\mathrm{E})$ which is topologically isomorphic to E. Let P be a continuous linear projection from $c_0(\mathrm{E})$ onto F.

Let us prove that E is bornological. As E is a Mackey space, it is enough to show that every locally bounded linear functional f' on F is continuous. But then $f' \circ \mathrm{P}$ is a locally bounded linear functional on $c_0(\mathrm{E})$, hence is continuous on $c_0(\mathrm{E})$ endowed with the Mackey topology which is sufficient since f' and $f' \circ \mathrm{P}$ coincide on F.

Let us prove the second condition. As E is bornological, every bounded subset of E'_β is equicontinuous. Therefore we get, by proposition 1.4 of [2], that $c_0(\mathrm{E})'$ is the space of the absolutely summable sequences in E'_B for some $\sigma(\mathrm{E}', \mathrm{E})$-closed absolutely convex and bounded subset B of E'_β and, by proposition 1.6 of [2], that $c_0(\mathrm{E})'$ is a sequentially dense subspace of $l^1(\mathrm{E}'_\beta)$. Moreover by proposition 2.1 of [2], $c_0(\mathrm{E})'_\beta$ is a topological subspace of $l^1(\mathrm{E}'_\beta)$. Hence $c_0(\mathrm{E})'$ coincides with $l^1(\mathrm{E}'_\beta)$ since $c_0(\mathrm{E})'_\beta$ is complete as strong dual of a bornological space.

(b) $\Rightarrow$ (a). As E is bornological, by use of propositions 1.4 and 2.1 of [2], we get as above that $c_0(\mathrm{E})'_\beta$ is topologically isomorphic to $l^1(\mathrm{E}'_\beta)$, hence is complete.

To show that $c_0(\mathrm{E})$ endowed with the Mackey topology is bornological, it is then enough to prove that every locally bounded linear functional $\underline{e}'$ on $c_0(\mathrm{E})$ belongs to $l^1(\mathrm{E}'_\beta)$.

For every $n \in \mathbb{N}$, the linear functional e'_n defined on E by

$$\langle e, e'_n \rangle = \langle e\varepsilon_n, \underline{e}' \rangle, \forall\, e \in \mathrm{E},$$

where we have set

$$e\varepsilon_n = (\underbrace{0, \ldots, 0, e}_{n}, 0, \ldots),$$

is certainly linear and locally bounded, hence continuous since E is bornological.

Let us show that the sequence e'_n belongs to $l^1(\mathrm{E}'_\beta)$. This is the case if we prove that for every absolutely convex and bounded subset B of E, we have

$$\sum_{n=1}^{\infty} p_{\mathrm{B}^0}(e'_n) < \infty.$$

For every $\varepsilon > 0$ and every $n, k \in \mathbb{N}$ such that $n \leqslant k$, there is of course $e_n^{(k)} \in \mathrm{B}$ such that

$$\langle e_n^{(k)}, e'_n \rangle + \frac{\varepsilon}{k} \geqslant p_{\mathrm{B}^0}(e'_n).$$

Therefore for every $\varepsilon > 0$ and every $k \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{n=1}^{k} p_{\mathrm{B}^0}(e'_n); &\leqslant \sum_{n=1}^{k} \langle e_n^{(k)}, e'_n \rangle + \varepsilon \\ &\leqslant \langle \sum_{n=1}^{k} e_n^{(k)}\, \varepsilon_n, \underline{e}' \rangle + \varepsilon \\ &\leqslant \sup_{\underline{e} \in \underline{B}} |\langle \underline{e}, \underline{e}' \rangle| + \varepsilon \end{aligned}$$

where $\underline{\mathrm{B}}$ is the following bounded subset of $c_0(\mathrm{E})$

$$\underline{\mathrm{B}} = \{ \sum_{n=1}^{k} e_n \varepsilon_n : k \in \mathbb{N} \ \&\ e_n \in \mathrm{B}, \forall n \}. \qquad (*)$$

Hence the conclusion.

To conclude we have just then to prove that we have

$$\langle \underline{e}, \underline{e}' \rangle = \sum_{n=1}^{\infty} \langle e_n, e'_n \rangle, \forall \underline{e} \in c_0(\mathrm{E}).$$

By construction this equality holds of course true on the dense linear subspace $\mathrm{E}^{(\mathbb{N})}$ of $c_0(\mathrm{E})$. Moreover for every $\underline{e} \in c_0(\mathrm{E})$, by hypothesis, there is an absolutely convex bounded subset B of E such that $\| e_n \|_{\mathrm{B}} \to 0$. This implies that the sequence $\sum_{n=1}^{\mathrm{N}} e_n \varepsilon_n$ converges to $\underline{e}$ in $c_0(\mathrm{E})_{\underline{B}}$ where $\underline{\mathrm{B}}$ is defined by (*), which implies

$$\sum_{n=1}^{\mathrm{N}} \langle e_n, e'_n \rangle = \langle \sum_{n=1}^{\mathrm{N}} e_n \varepsilon_n, \underline{e}' \rangle \to \langle \underline{e}, e' \rangle$$

since $\underline{e}'$ is locally bounded and linear.

THEOREM 2. *If every converging sequence of* E *is Mackey converging, then* $c_0(E)$ *is bornological if and only if* E *is bornological and such that* E'_β *has property* (B).

Proof. The condition is necessary. On one hand it is known that E is topologically isomorphic to a complemented subspace of $c_0(E)$, so E is bornological. On the other hand E'_β has property (B) as a direct consequence of the Marquina-Sanz Serna theorem.

The condition is sufficient. By the Marquina-Sanz Serna theorem, $c_0(E)$ is infrabarrelled, hence is a Mackey space. The conclusion follows then at once from theorem 1.

THEOREM 3. *If* E *is bornological, if every converging sequence of* E *is fast converging and if* E'_β *has property* (B), *then* $c_0(E)$ *is ultrabornological.*

Proof. By hypothesis, the converging, the Mackey converging and the fast converging sequences of coïncide. In this way, we know that $c_0(E)$ is bornological and that $c_0(E)'_{fc}$ is equal to $c_0(E)'_{mc}$ which is complete, hence the conclusion.

Remark : $E = \mathbb{R}^{\mathbb{R}}$ is an example of an ultrabornological space where there are converging sequences which are not Mackey converging, and such that however $c_0(\mathbb{R}^{\mathbb{R}})$ is ultrabornological since it is topologically isomorphic to $(c_0)^{\mathbb{R}}$.

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