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ELECTROMAGNETIC FIELDS AND POTENTIALS INVARIANT UNDER E(3) AND ITS SUBGROUPS

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ABSTRACT

Physical constraints (Maxwell theory) are imposed on invariant skewsymmetric tensors of order 2 and on invariant four-vectors but when E(3)-subsymmetries are taken into consideration. The corresponding physical fields *and* potentials are summarized in Tables and their connections are discussed.

1. INTRODUCTION

Invariance under the conformal and Poincaré groups or under their subgroups has recently been applied to different kinds of *tensor* and *spinor fields* [1,10]. More precisely, let us mention that necessary and sufficient conditions of *invariance on four-vectors* under the (connected) Poincaré group have been established and applied [8] when the considered Poincaré subgroup is the Euclidean group in three dimensions E(3). Such a Poincaré subgroup is an interesting one in connection, for example, with nonrelativistic quantum mechanics. Indeed, E(3) is the symmetry group of the free time-independent Schrödinger equation; its subgroups have already been studied [11] in connection with explicit symmetry breaking terms associated with scalar (V) and vector ($\vec{A}$) « potentials ».

As everybody knows, the scalar and vector « potentials » do form a four-vector in the context of restricted relativity. Such a remark is sufficient in order to justify an interrelation between the studies of invariant four-vectors [8] and the explicit symmetry breakings [11] when E(3) is the general symmetry required for a *free* particle description. Parts of this program are included in the work of Beckers and Hussin [8] but they do not consider *physical* four-vectors, i.e. electromagnetic four-potentials which have to be solutions of Maxwell theory. The aim of this note is precisely to add physical informations to these contributions and, moreover, to learn more deeply what are the connections between invariant electromagnetic fields and potentials. Such a study can also give a better understanding of minimal electromagnetic coupling schemes as those discussed by Combe and Richard [12], Hoogland [13], and Hussin [14]. The so-called compensating gauge transformations [15 16] can finally be exploited in such a context in order to give a precise meaning of the symmetry groups of the four-potentials A.

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The contents of this note are distributed as follows : in Section 2 we recall the notations and the essential results of Beckers and Hussin [8] when invariance conditions under E(3) and its subgroups are expressed on vectors $\vec{X} \equiv \vec{E}, \vec{B}, \vec{A}$ and on scalars V. The results are summarized in Table I. Section 3 deals with invariant electromagnetic fields and potentials when Maxwell theory is taken into account outside the charge and current distributions : the invariant fields (Table II) and potentials (Table III) are explicitly given when invariance under the ten E(3)-subgroups is considered. Evidently, only static fields are concerned in the case of E(3) and its subgroups. The explicit case of the $\overline{SO(2)}$-invariance is treated as an example in the Appendix A and the Maxwell $\vec{E}, \vec{B}, \vec{A}$-vectors and V-scalars are completely determined. In section 4, we discuss how to use the Tables II and III and their interest in connection with physical theories. Through an explicit example (T(2)-invariance) we give different informations on connections between invariant fields and their associated (invariant or not) potentials. Compensating gauge transformations are also mentioned and can be easily calculated.

2. VECTORS AND SCALARS INVARIANT UNDER THE E(3)-SUBGROUPS

In order to relate this contribution with the work of Beckers and Hussin [8], let us recall some definitions and conventions in the context of the symmetry group E(3) of Euclidean space in three (spatial) dimensions and its subsymmetries. The group E(3) corresponds to the set of transformations $(\vec{a}, R)$:

$$\vec{x} \to \vec{x}' = R\vec{x} + \vec{a} \tag{2.1}$$

leaving invariant the Euclidean distance between two points of the three-dimensional *real* vector space. Its Lie algebra is generated by six operators : three (infinitesimal) spatial translations P^i $(i = 1, 2, 3)$ associated with the parameters a^i and three (infinitesimal) spatial rotations J^i associated with the parameters θ^i so that its commutation relations are :

$$[P^i, P^k] = 0,\ [J^i, J^k] = i\varepsilon^{ikl}J^l,\ [J^i, P^k] = i\varepsilon^{ikl}P^l \tag{2.2}$$

As already recalled [8], the E(3)-subgroup structure is well-known; there are *ten* nontrivial and nonequivalent subgroups

$$
\begin{aligned}
&— T(1) \equiv \{P^3\},\ SO(2) \equiv \{J^3\},\ \overline{SO(2)} \equiv \{J^3 + aP^3, a \neq 0\},\\
&— T(2) \equiv \{P^1, P^2\},\ SO(2) \otimes T(1) \equiv \{J^3, P^3\},\\
&— T(3) \equiv \{P^1, P^2, P^3\},\ SO(3) \equiv \{\vec{J}\},\ E(2) \equiv \{J^3, P^1, P^2\},\\
&\quad \overline{E(2)} \equiv \{J^3 + aP^3, P^1, P^2, a \neq 0\},\\
&— E(2) \otimes T(1) \equiv \{J^3, \vec{P}\}
\end{aligned}
\tag{2.3}
$$

More precisely the $\overline{SO(2)}$- and $\overline{E(2)}$-cases correspond to infinite families parametrized by the nonzero real quantity a.

As particular cases of more general studies [2,4] the invariance conditions on skewsymmetric tensors F of order two and on four-vectors A can immediately be written in the E(3)-context. With the usual notations $F \equiv (\vec{E}, \vec{B})$ and $A \equiv (V, \vec{A})$, these relations are of the following form :

$$\vec{\theta} \wedge \vec{X} + D\vec{X} = 0,\ (\vec{X} = \vec{E}, \vec{B}, \vec{A}) \tag{2.4}$$

$$\mathrm{D}\mathrm{V} = 0, \tag{2.5}$$

where

$$\mathrm{D} \equiv (\vec{r} \wedge \vec{\theta} - \vec{a}).\vec{\nabla}, \tag{2.6}$$

$$\vec{\nabla} \equiv \{\partial_i\} \equiv \left(\frac{\partial}{\partial x^i}\right) \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{2.7}$$

Conditions (2.4) (when $\vec{\mathrm{X}} \equiv \vec{\mathrm{A}}$) and (2.5) have already been exploited [8] so that we can summarize these results in Table I. We give there the vectors $\vec{\mathrm{X}} \equiv \vec{\mathrm{A}}$, $\vec{\mathrm{E}}$ or $\vec{\mathrm{B}}$ and scalars V invariant under each of the ten above mentioned subgroups of E(3). Their explicit dependence in terms of cartesian (x, y, z), cylindrical $\left(\rho = (x^2 + y^2)^{1/2}, \varphi = tg^{-1}\frac{y}{x}, z\right)$, helical $\left(\rho, u = \frac{1}{2a}(z + a\varphi), v = \frac{1}{2a}(z - a\varphi)\right)$ or spherical $(r = (x^2 + y^2 + z^2)^{1/2}, \theta, \varphi)$ coordinates is also given. The different C, C′ are arbitrary constants.

Let us notice that at the level of the quantities $\mathrm{A} \equiv (\mathrm{V}, \vec{\mathrm{A}})$, there is a *one-to-one* correspondence between the invariant A and the E(3)-subgroups but not at the level of only invariant scalars $\mathrm{A}^0 \equiv \mathrm{V}$. Such a remark is directly connected with another approach [11] when symmetry breaking terms have been introduced in the time-independent Schrödinger equation. Finally, let us add that this one-to-one correspondence is *already* effective at the level of invariant *3-vectors* $\vec{\mathrm{X}} \equiv \vec{\mathrm{E}}, \vec{\mathrm{B}}$ or $\vec{\mathrm{A}}$.

3. INVARIANT PHYSICAL FIELDS AND POTENTIALS

In Section 2, physical informations on « fields » and « potentials » have really not been included. In order to get physical results, we have to give to F and A the meaning of an *electromagnetic* tensor and *electromagnetic* four-potential respectively by requiring that F and A are solutions of the Maxwell theory. In fact, when time independence is imposed, the physical tensor $\mathrm{F} \equiv (\vec{\mathrm{E}}, \vec{\mathrm{B}})$ has to be solution of the following set of Maxwell equations (in Gaussian units) :

$$\mathrm{rot}\,\vec{\mathrm{E}} = 0,\ \mathrm{div}\,\vec{\mathrm{E}} = 4\pi\rho, \tag{3.1}$$

$$\mathrm{rot}\,\vec{\mathrm{B}} = \frac{4\pi}{c}\vec{j},\ \mathrm{div}\,\vec{\mathrm{B}} = 0$$

and the four-potential $\mathrm{A} \equiv (\mathrm{V}, \vec{\mathrm{A}})$ has to satisfy the equations

$$\Delta\mathrm{V} = -4\pi\rho, \tag{3.2}$$

$$\Delta\vec{\mathrm{A}} = -\frac{4\pi}{c}\vec{j}.$$

The problem we have now to solve is the following : « what is the form of explicit solutions of Eqs. (3.1) or (3.2) invariant under the different E(3) subgroups ? ». Such a problem is directly related to the explicitly given sources (ρ and $\vec{j}$) and is difficult in general. We shall determine here static fields outside charge and current distributions. Consequently, these cases correspond to solve the following Maxwell equations on the fields $\vec{\mathrm{E}}, \vec{\mathrm{B}}$:

TABLE I

Invariant « Fields » and « Potentials »

Subgroups	Invariant vectors $\vec{X} = \vec{E}, \vec{B}, \vec{A}$	Invariant scalars
1. T(1)	$\vec{X} = \vec{X}(x, y)$	$V = V(x, y)$
2. SO(2)	$X^1 = i\,[f(\rho, z)\, e^{i\varphi} - g(\rho, z)\, e^{-i\varphi}]$ $X^2 = f(\rho, z)\, e^{i\varphi} + g(\rho, z)\, e^{-i\varphi}$ $X^3 = h(\rho, z)$	$V = V(\rho, z)$
3. $\overline{SO(2)}$	$X^1 = i\,[f(\rho, v)\, e^{iu} - g(\rho, v)\, e^{-iu}]$ $X^2 = f(\rho, v)\, e^{iu} + g(\rho, v)\, e^{-iu}$ $X^3 = h(\rho, v)$	$V = V(\rho, v)$
4. T(2)	$\vec{X} = \vec{X}(z)$	$V = V(z)$
5. SO(2) ⊗ T(1)	$X^1 = i\,[f(\rho)\, e^{i\varphi} - g(\rho)\, e^{-i\varphi}]$ $X^2 = f(\rho)\, e^{i\varphi} + g(\rho)\, e^{-i\varphi}$ $X^3 = h(\rho)$	$V = V(\rho)$
6. SO(3)	$\vec{X} = f(r)\, \dfrac{\vec{r}}{r}$	$V = V(r)$
7. E(2)	$X^1 = 0$ $X^2 = 0$ $X^3 = X(z)$	$V = V(z)$
8. T(3)	$\vec{X} = \vec{C}$	$V = C_1$
9. $\overline{E(2)}$	$X^1 = i\,[Ce^{i(u+v)} - C'e^{-i(u+v)}]$ $X^2 = Ce^{i(u+v)} + C'e^{-i(u+v)}$ $X^3 = C$	$V = C_1$
10. E(2) ⊗ T(1)	$X^1 = 0$ $X^2 = 0$ $X^3 = C$	$V = C_1$

$$\text{rot}\, \vec{X} = 0, \ \text{div}\, \vec{X} = 0, \ \vec{X} = \vec{E}, \vec{B} \tag{3.3}$$

or on the potentials $V, \vec{A}$:

$$\Delta V = 0, \quad \Delta \vec{A} = 0 \tag{3.4}$$

So, at this stage, what we shall call the « invariant physical fields » are the invariant fields of Table I submitted to the conditions (3.3) *and* the « invariant physical potentials » are the invariant vectors $\vec{A}$ and scalar V of Table I submitted to the conditions (3.4). All the results are summarized in Tables II and III respectively. Letters without explicit dependence are arbitrary constants.

TABLE II

Invariant Fields

Subgroups	Fields $\vec{X} = \vec{E}, \vec{B}$
1. T(1)	$X^1 = (C_1 e^{iky} + C_2 e^{-iky})(C_3 e^{kx} + C_4 e^{-kx})$ $X^2 = i(C_1 e^{iky} - C_2 e^{-iky})(C_3 e^{kx} - C_4 e^{-kx})$ $X^3 = C$
2. SO(2)	$X^1 = i[J_1(k\rho)\{C_1 e^{kz} + C_2 e^{-kz}\} e^{i\varphi} + J_1(k\rho)\{C_1 e^{kz} + C_2 e^{-kz}\} e^{-i\varphi} - \{C_3\rho + \frac{C_4}{\rho}\} e^{-i\varphi}]$ $X^2 = J_1(k\rho)\{C_1 e^{kz} + C_2 e^{-kz}\} e^{i\varphi} - J_1(k\rho)\{C_1 e^{kz} + C_2 e^{-kz}\} e^{-i\varphi} + \{C_3\rho + \frac{C_4}{\rho}\} e^{-i\varphi}$ $X^3 = -2i(C_1 e^{kz} - C_2 e^{-kz}) J_0(k\rho)$
3. $\overline{SO(2)}$	$X^1 = i\left[\frac{C_1}{\rho} e^{i(u-v)} - \frac{C_2}{\rho} e^{-i(u-v)}\right]$ $X^2 = \frac{C_1}{\rho} e^{i(u-v)} + \frac{C_2}{\rho} e^{-i(u-v)}$ $X^3 = C$
4. T(2)	$\vec{X} = (C_1, C_2, C_3)$
5. SO(2) ⊗ T(1)	$X^1 = i\left[\frac{C_1}{\rho} e^{i\varphi} - \frac{C_2}{\rho} e^{-i\varphi}\right]$ $X^2 = \frac{C_1}{\rho} e^{i\varphi} + \frac{C_2}{\rho} e^{-i\varphi}$ $X^3 = C$
6. SO(3)	$\vec{X} = \frac{C}{r^2} \frac{\vec{r}}{r}$
7. E(2)	$\vec{X} = (0, 0, C)$
8. T(3)	$\vec{X} = (C_1, C_2, C_3)$
9. $\overline{E(2)}$	$\vec{X} = (0, 0, C)$
10. E(2) ⊗ T(1)	$\vec{X} = (0, 0, C)$

TABLE III

Invariant Potentials

Subgroups	Scalars V	Vectors $\vec{A}$
1. T(1)	$V = (C_1 e^{iky} + C_2 e^{-iky})(C_3 e^{kx} + C_4 e^{-kx})$	$A^j = (C_1^j e^{ik^j y} + C_1'^j e^{-ik^j y})(C_2^j e^{k^j x} + C_2'^j e^{-k^j x})$
2. SO(2)	$V = J_0(k\rho)\{C_1 e^{kz} + C_2 e^{-kz}\}$	$A^1 = i\,[J_1(k\rho)\{C_1 e^{kz} + C_2 e^{-kz}\} e^{i\varphi} - J_1(m\rho)\{C_3 e^{mz} + C_4 e^{-mz}\} e^{-i\varphi}]$ $A^2 = J_1(k\rho)\{C_1 e^{kz} + C_2 e^{-kz}\} e^{i\varphi} + J_1(m\rho)\{C_3 e^{mz} + C_4 e^{-mz}\} e^{-i\varphi}$ $A^3 = J_0(k'\rho)\{C_3 e^{k'z} + C_3 e^{-k'z}\}$
3. $\overline{SO(2)}$	$V = J_n^{mod}\left(\frac{k\rho}{2a}\right)\{d_1 e^{ikv} + d_2 e^{-ikv}\}$	$A^1 = i\,[\left(h_1\rho + \frac{h_2}{\rho}\right) e^{i(u-v)} - \left(h_3\rho + \frac{h_4}{\rho}\right) e^{-i(u-v)}]$ $A^2 = \left(h_1\rho + \frac{h_2}{\rho}\right) e^{i(u-v)} + \left(h_3\rho + \frac{h_4}{\rho}\right) e^{-i(u-v)}$ $A^3 = J_n^{mod}\left(\frac{k'\rho}{2a}\right)\{d_3 e^{ik'v} + d_4 e^{-ik'v}\}$
4. T(2)	$V = C_1 z + C_2$	$A^1 = a^i z + b^i \ (i = 1, 2, 3)$
5. SO(2) ⊗ T(1)	$V = C_1 ln\rho + C_2$	$A^1 = i\,[\left(C_1\rho + \frac{C_2}{\rho}\right) e^{i\varphi} - \left(C_3\rho + \frac{C_4}{\rho}\right) e^{-i\varphi}]$ $A^2 = \left(C_1\rho + \frac{C_2}{\rho}\right) e^{i\varphi} + \left(C_3\rho + \frac{C_4}{\rho}\right) e^{-i\varphi}$ $A^3 = C_1' ln\rho + C_2'$
6. SO(3)	$V = \frac{C_1}{r} + C_2$	$\vec{A} = \frac{C_1'}{r}\frac{\vec{r}}{r} + C_2' r^2 \frac{\vec{r}}{r}$
7. E(2)	$V = C_1 z + C_2$	$\vec{A} = (0, 0, C_1' z + C_2')$
8. T(3)	$V = C$	$\vec{A} = (C_1, C_2, C_3)$
9. $\overline{E(2)}$	$V = C$	$\vec{A} = (0, 0, C')$
10. E(2) ⊗ T(1)	$V = C$	$\vec{A} = (0, 0, C')$

The explicit calculations are sometimes simple or sophisticated depending on the subgroup under consideration. For brevity, as an illustration, we only give in the Appendix a complete treatment of a significative case : the $\overline{\mathrm{SO}(2)}$-subgroup and the determination of $\vec{E}$, $\vec{B}$, V and $\vec{A}$.

In the Tables II and III enter relatively simple coordinate dependences and the only functions on which we have to give more informations are those denoted by J_n and J_n^{mod} ($n = 0, 1, 2, \dots$) : these are respectively the usual and modified Bessel functions [17]. The different calculations do use generalities on partial differential equations and we especially refer to Webster's standard book [18]. In particular, in order to be as complete as possible, let us mention that we often use the fact that the radial part of the Laplace equation can always be written as :

$$\Delta V = \frac{1}{r^{n-1}} \frac{d}{dr} \left(r^{n-1} \frac{dV}{dr} \right) = 0, \tag{3.5}$$

when the Euclidean space is of dimension n. Finally, let us add a general comment on *constant* and *uniform* fields such as those obtained in Table II (cf. cases 4, 7, 8, 9 and 10) : such fields can be directly deduced from the so-called *symmetric gauge* [15,16] corresponding to the relativistic writing ($F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$) :

$$A_\mu = \frac{1}{2} F_{\mu\rho} x^\rho \ (\mu = 0, 1, 2, 3) \tag{3.6}$$

when $F^{0i} = E^i$, $B^i = \frac{1}{2} \varepsilon^{ijk} F_{jk}$, $\varepsilon^{123} = 1$.

In terms of $(V, \vec{A})$ and $(\vec{E}, \vec{B})$ corresponding to a time independent theory, we have [19] :

$$V = -\vec{E} . \vec{r}, \quad \vec{A} = \frac{1}{2} \vec{B} \wedge \vec{r}. \tag{3.7}$$

Let us also notice that *regularity* conditions (at infinity, for example) can be imposed on the potentials V and $\vec{A}$: then we get informations on the arbitrary constants entering in Table III.

Before studying the connections between our results contained in Tables II *and* III, let us consider more particularly some of the physical fields given in Table II. The reason is, as everybody knows, that the fundamental physical entities are fields, whereas potentials fall into equivalence classes through gauge transformations. Then field symmetry is more interesting and our solutions show specific physical peculiarities in some cases. For example when SO(3)-invariance is required, we recover the fields generated by electric *or* magnetic monopoles (remember that $\vec{X} \equiv \vec{E}$ *or* $\vec{B}$) when, as it is in our case, we study the Maxwell theory *outside* the charge distributions. Moreover we notice that the smallest required invariance in order to get *constant physical* fields is the T(2)-invariance, a result which was unexpected before our study. Let us also remark that this T(2)-invariance is associated with charge and current distributions on an infinite plane (which can be seen as the limit of a slab of finite thickness). More generally, let us mention that each potential and its associated field can easily be put in correspondence with specific charge and current distributions : for example, the SO(2) $\otimes$ T(1) case is associated with cylindrical distributions, etc ..

4. HOW TO USE THE RESULTS OF TABLES II AND III?

From Tables II and III there are different ways to exploit the results and to extract some conclusions. *First*, let us notice that, at the level of « invariant physical fields », we loose the one-to-one correspondence with the different subgroups (cf. Table II) : the cases *4* and 8 lead to the same field as also the cases 7, 9 *and* 10 give another one. From Table III, the correspondence is still one-to-one at the level of « invariant physical potentials » except for the case 9 and 10. From Tables II and III we also remark that the vector potentials $\vec{A}$ have more general forms than the field vectors $\vec{E}$ and $\vec{B}$: this was expected because solutions of $\Delta\vec{A} = 0$ are more general than those of Eqs. (3.3). The last ones appear as particular because let us remember that

$$\Delta\vec{X} = \text{grad div } \vec{X} - \text{rot rot } \vec{X}. \tag{4.1}$$

Such differences disappear when the symmetries become stronger and stronger as it directly appears from the Tables (cf. cases 8, 9, 10).

Secondly, let us use elementary vector analysis in order to simplify and exploit the results of Tables II and III. We know the two following properties :

i) a null (magnetic) field $\vec{B}$ can always be written in the form :

$$\vec{B} = \text{rot } \vec{A} = 0 \Rightarrow \vec{A} = \text{grad } \varphi \tag{4.2}$$

where φ is a scalar space-dependent function;

ii) an indivergential (magnetic) field $\vec{B}$ can always be derived from a vector potential $\vec{A}$ with one component identical to zero :

$$\text{div } \vec{B} = 0 \rightarrow \vec{B} = \text{rot } \vec{A} \text{ with } \vec{A} \equiv (A^1, A^2, 0). \tag{4.3}$$

The simultaneous use of the above properties can make clear the connections between invariant fields and potentials. Let us illustrate these considerations on a specific example : T(2)-*invariance*.

a) From Table II, we have the *invariant fields* :

$$\vec{E} = (C_1, C_2, C_3), \ \vec{B} = (D_1, D_2, D_3) \tag{4.4}$$

where the C_i and D_i ($i = 1, 2, 3$) are constants. From the general remark (3.7), we can easily determine the potentials V and $\vec{A}$ leading to the fields (4.4). These are :

$$V = -(C_1x + C_2y + C_3z) + C_4, \tag{4.5}$$

$$\vec{A} = \frac{1}{2}(D_2z - D_3y + D_4, D_3x - D_1z + D_5, D_1y - D_2x + D_6), \tag{4.6}$$

where the constants C_4, D_4, D_5 and D_6 are added in all generality;

b) From Table III, T(2)-invariance says that the *invariant potentials* are :

$$V = c_1z + c_2, \tag{4.7}$$

$$\vec{A} \equiv (A^i), \ A^i = a^iz + b^i, \ (i = 1, 2, 3), \tag{4.8}$$

leading to the *physical* fields :

$$\vec{E} = (0, 0, -c_1),\ \vec{B} = (-a^2, a^1, 0). \tag{4.9}$$

So they coïncide with the invariant fields (4.4) when

$$\begin{aligned} C_1 &= 0,\ C_2 = 0,\ C_3 = -c_1, \\ D_1 &= -a^2,\ D_2 = a^1,\ D_3 = 0; \end{aligned} \tag{4.10}$$

c) Let us now combine the implications of the two Tables. We notice that the (nonnecessarily invariant) potentials given by Eqs. (4.5) and (4.6) *and* leading to the (invariant) fields (4.9) (i.e. (4.4) with (4.10)) have the following form :

$$V = -C_3 z + C_4 \tag{4.11}$$

$$\vec{A} = \frac{1}{2}(D_2 z + D_4, -D_1 z + D_5, D_1 y - D_2 x + D_6). \tag{4.12}$$

If Eqs. (4.7) and (4.11) are identical, we point out that there are differences in the forms (4.8) and (4.12) of the vector potential $\vec{A}$. So let us impose a *gauge* transformation on $\vec{A}$:

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{A}_0 \tag{4.13}$$

such that through the properties (4.2) and (4.3) we have

$$\vec{B}_0 = \text{rot}\,\vec{A}_0 = 0 \to \vec{A}_0 = \text{grad}\,\varphi = (A_0^1, A_0^2, A_0^3) \tag{4.14}$$

and

$$\text{div}\,\vec{B} = 0 \to \vec{B} = \text{rot}\,\vec{A}' = \text{rot}\,\vec{A} \text{ with } \vec{A}' = (A'^1, A'^2, 0). \tag{4.15}$$

With $\vec{A}$ given by Eq. (4.12), we choose :

$$A_0^3 = \frac{1}{2}(-D_1 y + D_2 x - D_6) \tag{4.16}$$

ensuring that Eq. (4.15) is satisfied. The simple implications of Eqs. (4.14) and (4.16) lead us to

$$\vec{A}_0 = \frac{1}{2}\left(D_2 z + 2\int \frac{\partial}{\partial x} f(x, y) dy, -D_1 z + 2f(x, y), -D_1 y + D_2 x - D_6\right). \tag{4.17}$$

So we finally get :

$$\begin{aligned} \vec{A}' &= \left(D_2 z + \frac{1}{2} D_4 + \int \frac{\partial}{\partial x} f(x, y) dy, -D_1 z + \frac{1}{2} D_5 + f(x, y), 0\right) \\ &= (D_2 z + D_4, -D_1 z + D_5, a^3 z + b^3) + \left(\int \frac{\partial}{\partial x} f(x, y) dy - \frac{1}{2} D_4, f(x, y) \right. \\ &\qquad \left. - \frac{1}{2} D_5, -(a^3 z + b^3)\right) \qquad (4.18) \\ &= \vec{A}_{\text{inv}}(z) + \text{grad}\,\chi(x, y, z) \qquad (4.19) \end{aligned}$$

when $\vec{A}_{\text{inv}}$ is given by Eq. (4.8) with

$$D_1 = -a^2,\ D_2 = a^1,\ D_4 = b^1,\ D_5 = b^2. \tag{4.20}$$

Eq. (4.19) shows the connection between the (nonnecessarily invariant) vector potential $\vec{A} \equiv$ (4.12) leading to particular invariant fields and the invariant vector potential $\vec{A}_{\text{inv}} \equiv$ (4.8). It illustrates their physical equivalence up to a gauge transformation fixed by the function $\chi(x, y, z)$ easily calculated from (4.18) and (4.19).

Such an approach can be related to the study of invariant fields and potentials through minimal coupling schemes as those discussed by Combe-Richard [12], Hoogland [13] and more recently by Hussin [14] applying *Poincaré* considerations. The constant field (4.9) can be associated with the *orthogonal* case (choose $-c_1 = a^1 = D_2 = C_3 = E$, $D_1 = -a^2 = 0$) : in that case we know [1] that its kinematical group is $G \equiv \{P^\mu, J^2 + K^3, J^3 - K^2\}$, isomorphic to the group G_2 as mentioned in Hussin's work [14]. Now, in our context of E(3)-symmetry, the kinematical group reduces to $\{\vec{P}\} \equiv T(3)$ as a consequence of time-independence and it is easy to show that the corresponding potential admits a subsymmetry as expected. In fact, if we search for the (orthogonal) field :

$$\vec{E} = (0, 0, E) \quad \text{and} \quad \vec{B} = (0, E, 0), \tag{4.21}$$

the corresponding invariant potential has the essential form

$$V = -Ez, \quad \vec{A} = (Ez, 0, 0) \tag{4.22}$$

and its symmetry group becomes the set $\{P_1, P_2\} \equiv T(2)$ as it can be deduced from Eqs. (2.4) and (2.5). From Table III, this result was evidently expected.

Thirdly, let us notice that we also get here ad-hoc results in order to illustrate the meaning of *compensating gauge transformations* as developed by Janner and Janssen [15]. If we limit ourselves once again to T(2)-invariance on $(\vec{E}, \vec{B})$, it is very easy to calculate the compensating gauge for $V \equiv$ (4.5) and $\vec{A} \equiv$ (4.6) according to Janner-Janssen notations :

$$gA - A = \partial\chi_g, \quad \forall g \in T(2), \tag{4.23}$$

when

$$x \xrightarrow{\;g\;} x' = x + t_1 \tag{4.24}$$

$$y \xrightarrow{\;g\;} y' = y + t_2.$$

We want to express our hearty thanks to Professor J. Beckers for stimulating discussions.

APPENDIX : INVARIANCE UNDER $\overline{SO(2)} \equiv (J^3 + aP^3;\ a \neq 0)$

Table I says that under $\overline{SO(2)}$ the invariant vectors $\vec{X} = \vec{E}, \vec{B}, \vec{A}$ are of the form

$$\begin{aligned} X^1 &= i[f(\rho, v)e^{iu} - g(\rho, v)e^{-iu}], \\ X^2 &= f(\rho, v)e^{iu} + g(\rho, v)e^{-iu}, \\ X^3 &= h(\rho, v), \end{aligned} \tag{A.1}$$

and the invariant scalar V is given by

$$V = V(\rho, v), \tag{A.2}$$

where (ρ, u, v) are *helical* coordinates :

$$\rho = (x^2 + y^2)^{1/2}, \; u = \frac{1}{2a}(z + a\varphi), \; v = \frac{1}{2a}(z - a\varphi) \tag{A.3}$$

well adapted for the treatment of $(J^3 + aP^3)$-invariance.

a) *Physical fields invariant under* $\overline{SO(2)}$.

Outside the source distributions in the Maxwell theory, the vectors $\vec{X} = \vec{E}, \vec{B}$ have to satisfy the following equations :

$$\begin{aligned} \text{rot}\, \vec{X} &= 0 \\ \text{div}\, \vec{X} &= 0 \end{aligned} \tag{A.4}$$

in order to describe physical field.

Let us put in (A.4) the values of $\vec{X} = $ (A.1). Combining the first two components of rot $\vec{X} = 0$, we obtain :

$$\left[\frac{1}{2\rho i}\frac{\partial}{\partial v} h(\rho, v) + \frac{\partial}{\partial \rho} h(\rho, v)\right] e^{-iv} = \frac{1}{a}\left(i\frac{\partial}{\partial v} - 1\right) f(\rho, v) \tag{A.5}$$

and

$$\left[\frac{1}{2\rho i}\frac{\partial}{\partial v} h(\rho, v) - \frac{\partial}{\partial \rho} h(\rho, v)\right] e^{iv} = \frac{1}{a}\left(i\frac{\partial}{\partial v} + 1\right) g(\rho, v) \tag{A.6}$$

from which it follows that :

$$\frac{\partial}{\partial v} h(\rho, v) = \frac{i\rho}{a}\{[\left(i\frac{\partial}{\partial v} - 1\right) f] e^{iv} + [\left(i\frac{\partial}{\partial v} + 1\right) g] e^{-iv}\}. \tag{A.7}$$

The third component of rot $\vec{X} = 0$ gives :

$$\{\left[\frac{\partial}{\partial \rho} + \frac{1}{2\rho}\left(1 + i\frac{\partial}{\partial v}\right)\right] f\} e^{iv} + \{\left[\frac{\partial}{\partial \rho} + \frac{1}{2\rho}\left(1 - i\frac{\partial}{\partial v}\right)\right] g\} e^{-iv} = 0. \tag{A.8}$$

From div $\vec{X} = 0$, we obtain :

$$\frac{\partial}{\partial v} h(\rho, v) = -2ai\{\left[\frac{\partial}{\partial \rho} + \frac{1}{2\rho}\left(1 + i\frac{\partial}{\partial v}\right)\right] f\} e^{iv} + 2ai\{\left[\frac{\partial}{\partial \rho} + \frac{1}{2\rho}\left(1 - i\frac{\partial}{\partial v}\right)\right] g\} e^{-iv}. \tag{A.9}$$

Then Eqs. (A.7) and (A.9) give :

$$\{\left[i + \frac{\partial}{\partial v} - a^2\left(2i\rho\frac{\partial}{\partial \rho} + i - \frac{\partial}{\partial v}\right)\right] f\} e^{iv} = \{\left[i - \frac{\partial}{\partial v} - a^2\left(2i\rho\frac{\partial}{\partial \rho} + i + \frac{\partial}{\partial v}\right)\right] g\} e^{-iv}. \tag{A.10}$$

From Eqs. (A.9) and (A.10), we get :

$$-\left[\left(2\rho\frac{\partial}{\partial \rho} + 1 - i\frac{\partial}{\partial v}\right) g\right] e^{-iv} = \left[\left(2\rho\frac{\partial}{\partial \rho} + 1 + i\frac{\partial}{\partial v}\right) f\right] e^{iv}. \tag{A.11}$$

Let us now search for (particular) solutions of the form

$$f(\rho, v) = \rho^m e^{-iv}, \; g(\rho, v) = \rho^n e^{iv}. \tag{A.12}$$

It is easily verified that we must have $m = n = -1$ so that we find :

$$f(\rho, v) = \frac{C_1}{\rho} e^{-iv}, \; g(\rho, v) = \frac{C_2}{\rho} e^{iv}, \; c_1, c_2 = \text{constants.} \tag{A.13}$$

Furthermore, by (A.6), (A.7), and (A.13), one can verify that

$$X^3 = C = \text{constant.} \tag{A.14}$$

So the components of the physical fields $\vec{X} = \vec{E}, \vec{B}$ invariant under $\overline{SO(2)}$ are those given in Table II, namely :

$$\begin{aligned} X^1 &= i\left[\frac{C_1}{\rho} e^{i(u-v)} - \frac{C_2}{\rho} e^{-i(u-v)}\right], \\ X^2 &= \frac{C_1}{\rho} e^{i(u-v)} + \frac{C_2}{\rho} e^{-i(u-v)}, \\ X^3 &= C. \end{aligned} \tag{A.15}$$

b) *Scalar potential invariant under* $\overline{SO(2)}$.

If the invariant scalar V corresponds to a free electric field, then it satisfies the particular Laplace equation :

$$\Delta V = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{4\rho^2}\frac{\partial^2}{\partial v^2} + \frac{1}{4a^2}\frac{\partial^2}{\partial v^2}\right) V(\rho, v) = 0. \tag{A.16}$$

In order to solve this equation, let us make a separation of variables by putting

$$V = R(\rho)T(v). \tag{A.17}$$

We obtain the following equations :

$$\frac{d^2T}{dv^2} = -k^2 T \tag{A.18}$$

and

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \left(\frac{k^2}{4a^2} + \frac{k^2}{4\rho^2}\right) R = 0 \tag{A.19}$$

where k^2 is the constant of separation.

The general solution of (A.18) is

$$T = d_1 e^{ikv} + d_2 e^{-ikv}. \tag{A.20}$$

In order to solve Eq. (A.19), we put :

$$\frac{k^2}{4a^2} = m^2, \; \frac{k^2}{4} = n^2, \; m\rho = x \tag{A.21}$$

and finally obtain the equation :

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{n^2}{x^2}\right) R = 0. \tag{A.22}$$

The solutions of this equation are the so-called modified Bessel functions of order n

denoted by « J_n^{mod} » [17]. Thus, the electromagnetic scalar potential invariant under $\overline{SO(2)}$ takes the form :

$$V(\rho, v) = J_n^{mod}\left(\frac{k\rho}{2a}\right)(d_1 e^{ikv} + d_2 e^{-ikv}) \tag{A.23}$$

as given in Table III.

c) *Vector potential invariant under* $\overline{SO(2)}$.

If the vector potential $\vec{A}$ has to describe free magnetic field, it must satisfy :

$$\Delta\vec{A} = 0. \tag{A.24}$$

Using the second component of $\vec{A}$ given by (A.1), Eq. (A.24) implies that :

$$\{\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{4}\left(\frac{1}{\rho^2} + \frac{1}{a^2}\right)\left(1 - \frac{\partial^2}{\partial v^2}\right) - \frac{i}{2}\left(\frac{1}{\rho^2} - \frac{1}{a^2}\right)\frac{\partial}{\partial v}\}f = 0 \tag{A.25}$$

and

$$\{\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{4}\left(\frac{1}{\rho^2} + \frac{1}{a^2}\right)\left(1 - \frac{\partial^2}{\partial v^2}\right) + \frac{i}{2}\left(\frac{1}{\rho^2} - \frac{1}{a^2}\right)\frac{\partial}{\partial v}\}g = 0. \tag{A.26}$$

With a solution of the kind :

$$f(\rho, v) = \rho^m e^{-iv}, \tag{A.27}$$

Eq. (A.25) leads to the conditions :

$$m^2 - 1 = 0 \rightarrow m = \pm 1, \tag{A.28}$$

so that we get :

$$f(\rho, v) = \left(h_1\rho + \frac{h_2}{\rho}\right)e^{-iv}, \ (h_1, h_2 = \text{constants}). \tag{A.29}$$

In a similar way, one can find from Eq. (A.26) that :

$$g(\rho, v) = \left(h_3\rho + \frac{h_4}{\rho}\right)e^{iv}, \ (h_3, h_4 = \text{constants}). \tag{A.30}$$

Finally, let us notice that the case of the third component of $\vec{A}$ is exactly similar to that of V since $A^3 = h(\rho, v)$ has to verify $\Delta A^3 = 0$.

Thus we obtain the results collected in Table III, namely :

$$\begin{aligned} A^1 &= i[\left(h_1\rho + \frac{h_2}{\rho}\right)e^{i(u-v)} - \left(h_3\rho + \frac{h_4}{\rho}\right)e^{-i(u-v)}], \\ A^2 &= [\left(h_1\rho + \frac{h_2}{\rho}\right)e^{i(u-v)} + \left(h_3\rho + \frac{h_4}{\rho}\right)e^{-i(u-v)}], \\ A^3 &= J_n^{mod}\left(\frac{k'\rho}{2a}\right)\{d_3 e^{ik'v} + d_4 e^{-ik'v}\}. \end{aligned} \tag{A.31}$$

REFERENCES

[1] H. Bacry, Ph. Combe and J. L. Richard, *Il Nuovo Cimento*, A **67**, 267-299 (1970).

[2] Ph. Combe and P. Sorba, *Physica*, A **80**, 271-286 (1975).

[3] J. Beckers and G. Comte, *Bull. Soc. Roy. Sc. Lg.*, **45**, 279-287 (1976).

[4] J. Beckers, J. Harnad, M. Perroud and P. Winternitz, *J. Math. Phys.*, **19**, 2126-2153 (1978).

[5] J. Beckers and M. Jaminon, *Physica*, A **94**, 165-180 (1978).

[6] J. Beckers, M. Jaminon and J. Serpe, *J. Phys.*, A **12**, 1341-1356 (1979).

[7] J. Beckers, J. Harnad and P. Jasselette, *J. Math. Phys.*, **21**, 2491-2499 (1980).

[8] J. Beckers and V. Hussin, *Found. Phys.*, **11**, 881-893 (1981).

[9] J. Beckers and V. Hussin, *J. Phys.*, A **14**, 317-326 (1981).

[10] V. Hussin, *J. Phys.*, A **15**, 319-321 (1982).

[11] J. Beckers, J. Patera, M. Perroud and P. Winternitz, *J. Math. Phys.*, **18**, 72-83 (1977).

[12] Ph. Combe and J. L. Richard, *Physica*, **71**, 66-74 (1974).

[13] H Hoogland, *J Phys.*, A **11**, 797-804 (1978).

[14] V. Hussin, *Il Nuovo Cimento*, A **65**, 39-50 (1981); *Lett. Nuovo Cimento*, **32**, 181-185 (1981).

[15] A. Janner and T. Janssen, *Physica*, **53**, 1-27 (1971).

[16] N. Giovannini, *Physica*, A **87**, 546-568 (1977).

[17] J. D. Jackson, *Classical Electrodynamics* (Wiley, Second Edition), 1975.

[18] A. G. Webster, *Partial Differential Equations of Mathematical Physics* (Dover, Second Corrected Edition), 1955.

[19] L. Landau and E. Lifchitz, *Théorie des champs* (Editions Mir, Moscou), 1970.

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