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TWO CONGRUENCES ON DISTRIBUTIVE LATTICES

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RÉSUMÉ

Soient $\mathscr{L} = (L ; \vee, \wedge, 0)$ un lattis distributif avec zéro, S un $\wedge$-sous-demi-lattis de $\mathscr{L}$ et J un idéal de $\mathscr{L}$. Alors nous pouvons définir deux familles de congruences Ψ^S et R^J par :

$$(a, b) \in \Psi^S \underset{\text{Déf}}{\equiv} \text{il existe } s \in S \text{ tel que } a \wedge s = b \wedge s ;$$

$$(a, b) \in R^J \underset{\text{Déf}}{\equiv} a \wedge x \in J \text{ si et seulement si } b \wedge x \in J \; (\forall x \in L).$$

Ces congruences ont de multiples propriétés qui facilitent l'étude de la structure des idéaux premiers et premiers-minimaux de $\mathscr{L}$. En particulier, l'espace des idéaux premiers de $\mathscr{L}$ modulo Ψ^S est homéomorphe au sous-espace des idéaux premiers de $\mathscr{L}$ ne rencontrant pas S, chacun de ces espaces étant muni de la topologie habituelle. L'espace des idéaux premiers-minimaux de $\mathscr{L}$ est homéomorphe à l'espace des idéaux premiers-minimaux de $\mathscr{L}$ modulo $R^{(0)} = R$. Ce dernier résultat est le point de départ de quelques théorèmes relatifs à l'espace des idéaux premiers-minimaux de $\mathscr{L}$. Lorsque $\mathscr{L}$ modulo R est un lattis de Boole, le cas est particulièrement intéressant et quelques résultats nouveaux sont fournis ici.

INTRODUCTION

In the course of some work on the prime ideals of distributive lattices [7], two families of congruences appeared useful in illustrating the concepts under discussion. Neither of these congruences has been studied in much detail on distributive lattices, although both have appeared before in various forms, see [2], [4]. We shall define and discuss these congruences in some detail, relating them to the other work mentioned. Throughout this note, $\mathscr{L} = (L ; \vee, \wedge, 0)$ will be a *distributive lattice* with zero.

1. NOTATION AND DEFINITIONS

The empty set is always denoted $\square$. Lattices are always written in script (e. g. $\mathscr{L}$, $\mathscr{L}/R$) and the corresponding Roman letter (e. g., L, L/R) is used for the carrier (basic set). The set of all ideals of a lattice $\mathscr{L}$ is written $I(\mathscr{L})$ and forms a lattice $\mathscr{I}_{\mathscr{L}}$. The set of all prime ideals (resp. minimal prime ideals) of $\mathscr{L}$ is denoted $\mathscr{P}_{\mathscr{L}}$ (resp. $\mathscr{M}_{\mathscr{L}}$).

For A, B subsets of L, we write $(A : B)^* = \{x \in L : x \wedge a \in B \text{ for all } a \in A\}$.

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If $A = \{a\}$, we write $(a : B)^*$ and if $B = \{0\}$ we employ A^* instead of $(A : \{0\})^*$ and $(a)^*$ instead of $\{a\}^*$.

When $a \in L$, we write (a) for the principal ideal of $\mathscr{L}$ generated by a, and $[a]$ for the principal dual ideal of $\mathscr{L}$ generated by a.

A congruence θ^J on $\mathscr{L}$ with any given ideal J as a congruence class has been defined in [2] and is equivalent to

$$(a, b) \in \theta^J \underset{\text{Df}}{\equiv} a \vee j = b \vee j \text{ for some } j \in J.$$

The canonical epimorphism induced by θ^J will be written θ^J. It is proved in [2] that θ^J is the smallest congruence with J as (zero) congruence class. Similarly (but dually) one may define $\check{\theta}^F$ for a dual ideal F, having similar properties.

Maps are always written on the right of elements from the domain. Further details concerning lattices can be found in [1].

2. THE CONGRUENCE Ψ^S : GENERALITIES

In this section, $S \subseteq L$ will be an arbitrary $\wedge$-subsemilattice of $\mathscr{L} = (L; \vee, \wedge, 0)$. We define Ψ^S by

$$(a, b) \in \Psi^S \underset{\text{Df}}{\equiv} a \wedge s = b \wedge s \text{ for some } s \in S.$$

PROPOSITION 2.1. — *The relation Ψ^S as defined above is a congruence on $\mathscr{L}$ and S is contained in a single congruence class.*

Proof. Ψ^S is readily seen to be reflexive and symmetric. If, for a, b, c in L we have $(a, b) \in \Psi^S$ and $(b, c) \in \Psi^S$, then there must exist s, s' in S with $a \wedge s = b \wedge s$ and $b \wedge s' = c \wedge s'$. Thus $a \wedge s \wedge s' = b \wedge s \wedge s' = c \wedge s \wedge c'$ and, since $s \wedge s' \in S$, we have proved that $(a, c) \in \Psi^S$ and so Ψ^S is an equivalence.

Take a, b, c, d in L with $(a, b) \in \Psi^S$ and $(c, d) \in \Psi^S$. Then $a \wedge s = b \wedge s$ for some $s \in S$ and $c \wedge s' = d \wedge s'$ for some $s' \in S$. From this it follows that $(a \vee c) \wedge (s \wedge s') = (b \vee d) \wedge (s \wedge s')$ and $(a \wedge c) \wedge (s \wedge s') = (b \wedge d) \wedge (s \wedge s')$ which proves that Ψ^S is a congruence.

Finally, if $s, s' \in S$ then $s \wedge (s \wedge s') = s' \wedge (s \wedge s')$ which proves that $(s, s') \in \Psi^S$ and so S is contained in a single congruence class.

NOTES. (i) We will write $\mathscr{L}//S$ instead of $\mathscr{L}/\Psi^S$ for the homomorphic image of $\mathscr{L}$ and the epimorphism will be denoted ψ^S.

(ii) If S is a dual ideal of $\mathscr{L}$, Ψ^S coincides with the unique smallest congruence having S as congruence class, see [2].

PROPOSITION 2.2. — *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two lattices with $S \subseteq L_1$ and $T \subseteq L_2$ two $\wedge$-subsemilattices of $\mathscr{L}_1$, $\mathscr{L}_2$ respectively. If $\varphi : \mathscr{L}_1 \to \mathscr{L}_2$ is a morphism with $S\varphi \subseteq T$, then there is a unique morphism $\psi_{1,2}^{S,T}$ such that $\psi_1^S \circ \psi_{1,2}^{S,T} = \varphi \circ \psi_2^T$. Further, if φ is an epimorphism so also is $\psi_{1,2}^{S,T}$, and if φ is a monomorphism with $S\varphi = T$, then so also is $\psi_{1,2}^{S,T}$.*

Proof. For $a = x\psi_1^S \in L_1 // S$, define $\psi_{1,2}^{S,T}$ by

$$a\psi_{1,2}^{S,T} = x\varphi\psi_2^T \text{ where } x \in L_1.$$

Then $\psi_{1,2}^{S,T}$ is well-defined, for if $a = y\psi_1^S$ for $y \in L_1$, i. e. if $(x, y) \in \Psi_1^S$, there would be $s \in \mathrm{S}$ such that $x \wedge s = y \wedge s$. In this case ; $x\varphi \wedge s\varphi = y\varphi \wedge s\varphi$ and, since $s\varphi \in \mathrm{T}$, we would have $(x\varphi, y\varphi) \in \Psi_2^{\mathrm{T}}$ proving that the image of a under $\psi_{1,2}^{\mathrm{S,T}}$ is independent of the representative chosen. The fact that the mappings compose as asserted is immediate.

If φ is an epimorphism, $\psi_{1,2}^{\mathrm{S,T}}$ is also, since ψ_2^{T} is always an epimorphism. Suppose now that φ is a monomorphism, and $\mathrm{S}\varphi = \mathrm{T}$. Assume $a\psi_{1,2}^{\mathrm{S,T}} = b\psi_{1,2}^{\mathrm{S,T}}$ where $a = x\psi_1^{\mathrm{S}} \in \mathrm{L}_1 \,//\, \mathrm{S}$ and $b = y\psi_1^{\mathrm{S}} \in \mathrm{L}_1 \,//\, \mathrm{S}$, and $x, y \in \mathrm{L}_1$. Then (by definition) $x\varphi\psi_2^{\mathrm{T}} = y\varphi\psi_2^{\mathrm{T}}$ and so $(x\varphi, y\varphi) \in \Psi_2^{\mathrm{T}}$, i. e. $x\varphi \wedge t = y\varphi \wedge t$ for some $t \in \mathrm{T}$. If we write $t = s\varphi$ for $s \in \mathrm{S}$, we obtain $x\varphi \wedge s\varphi = y\varphi \wedge s\varphi$ or $(x \wedge s)\varphi = (y \wedge s)\varphi$ which implies that $x \wedge s = y \wedge s$ since φ is a monomorphism. Thus $(x, y) \in \Psi_2^{\mathrm{S}}$ and so $a = x\psi_1^{\mathrm{S}} = y\psi_1^{\mathrm{S}} = b$ proving that $\psi_{1,2}^{\mathrm{S,T}}$ is a monomorphism.

Proposition 2.3. — *Let* $\mathscr{L}$ *be a lattice and* S *a* $\wedge$*-subsemilattice of* $\mathscr{L}$. *If* $\mathrm{J}' \in \mathrm{I}(\mathscr{L} \,//\, \mathrm{S})$ *and* $\mathrm{J} = \mathrm{J}'(\psi^{\mathrm{S}})^{-1}$, *then* $\mathrm{J} \in \mathrm{I}(\mathscr{L})$ *and the canonical epimorphism* $\theta = \theta^{\mathrm{J}} : \mathscr{L} \to \mathscr{L}/\mathrm{J}$ *induces*

$$\hat{\theta} : \mathscr{L} \,//\, \mathrm{S} \to (\mathscr{L}/\mathrm{J}) \,//\, \mathrm{S}\theta.$$

The map $\hat{\theta}$ *has kernel* J′ *and defines a canonical isomorphism*

$$\hat{\hat{\theta}} : (\mathscr{L} \,//\, \mathrm{S})/\mathrm{J}' \overset{\simeq}{\to} (\mathscr{L}/\mathrm{J}) \,//\, \mathrm{S}\theta.$$

Proof. Since $\mathrm{J}' \in \mathrm{I}(\mathscr{L} \,//\, \mathrm{S})$ and ψ^{S} is an epimorphism, $\mathrm{J}'(\psi^{\mathrm{S}})^{-1}$ is always an ideal of $\mathscr{L}$. Now for $\hat{\theta}$. Consider the diagram

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\theta = \theta^{\mathrm{J}}} & \mathscr{L}/\mathrm{J} \\ \psi^{\mathrm{S}} \downarrow & & \downarrow \psi^{\mathrm{S}\theta} \\ \mathscr{L} \,//\, \mathrm{S} & \xrightarrow{\hat{\theta}} & (\mathscr{L}/\mathrm{J}) \,//\, \mathrm{S}\theta \end{array}$$

Proposition 2.2 above gives the existence of an induced map $\hat{\theta} = \psi_{\mathscr{L},\mathscr{L}/\mathrm{J}}^{\mathrm{S,S}\theta}$. We will look at the kernel of $\hat{\theta}$ (which exists since $\mathscr{L}/\mathrm{J}$, and so $(\mathscr{L}/\mathrm{J}) \,//\, \mathrm{S}\theta$, possesses a lattice zero). For $a \in \mathrm{L} \,//\, \mathrm{S}$, $a\hat{\theta} = 0_{(\mathscr{L}/\mathrm{J})//\mathrm{S}\theta}$ iff there is $x \in \mathrm{L}$ with $a = x\psi^{\mathrm{S}}$ and $x\theta^{\mathrm{J}}\psi^{\mathrm{S}\theta} = 0_{(\mathscr{L}/\mathrm{J})//\mathrm{S}\theta}$. But this means there is $s\theta \in \mathrm{S}\theta$ with $x\theta \wedge s\theta = 0_{\mathscr{L}/\mathrm{J}}$ which gives us $x \wedge s \in \mathrm{J}$ and so $(x \wedge s)\psi^{\mathrm{S}} \in \mathrm{J}'$. It is readily checked that $s\psi^{\mathrm{S}}$ behaves as the (lattice) unit of $\mathscr{L} \,//\, \mathrm{S}$ and so we deduce that $x\psi^{\mathrm{S}} = a \in \mathrm{J}'$. Thus the kernel of $\hat{\theta}$ is J′ as asserted.

Next consider the diagram

$$\begin{array}{ccc} \mathscr{L} \,//\, \mathrm{S} & \xrightarrow{\hat{\theta}} & (\mathscr{L}/\mathrm{J}) \,//\, \mathrm{S}\theta \\ & {\scriptstyle \theta^{\mathrm{J}'}} \searrow \quad \nearrow {\scriptstyle \hat{\hat{\theta}}} & \\ & (\mathscr{L} \,//\, \mathrm{S})/\mathrm{J}' & \end{array}$$

Define, for $a \in (\mathscr{L} \,//\, \mathrm{S})/\mathrm{J}'$, $a\hat{\hat{\theta}} = x\hat{\theta}$ where $a = x\theta^{\mathrm{J}'}$. It is straight-forward to check that $\hat{\hat{\theta}}$ is well-defined. We will show that $\hat{\hat{\theta}}$, obviously an epimorphism, is an isomorphism. Suppose $a\hat{\hat{\theta}} = b\hat{\hat{\theta}}$ where $a = x\theta^{\mathrm{J}'}$ and $b = y\theta^{\mathrm{J}'}$ for x, y in $\mathscr{L} \,//\, \mathrm{S}$. Then

$x = c\psi^S$ and $y = d\psi^S$ for c, d in L and $a\hat{\hat{\theta}} = b\hat{\hat{\theta}}$ means $x\hat{\theta} = y\hat{\theta}$ which is equivalent to $(c\theta^J, d\theta^J) \in \Psi^{S\theta}_{\mathscr{L}/J}$. Thus

$$(c\theta^J, d\theta^J) \in \Psi^{S\theta}_{\mathscr{L}/J}$$

$\Rightarrow c\theta^J \wedge k = d\theta^J \wedge k$ for some $k = s\theta^J \in S\theta^J$,

$\Rightarrow (c \wedge s)\theta^J = (d \wedge s)\theta^J$,

$\Rightarrow (c \wedge s, d \wedge s) \in \theta^J$,

$\Rightarrow (c \wedge s) \vee j = (d \wedge s) \vee j$ for some $j \in J$. Apply ψ^S :

$\Rightarrow [(c \wedge s) \vee j]\psi^S = [(d \wedge s) \vee j]\psi^S$,

$\Rightarrow (c\psi^S \wedge s\psi^S) \vee j\psi^S = (d\psi^S \wedge s\psi^S) \vee j\psi^S$,

$\Rightarrow c\psi^S \vee j\psi^S = d\psi^S \vee j\psi^S$,

$\Rightarrow x \vee j' = y \vee j'$ for $j' = j\psi^S \in J' = J\psi^S$,

$\Rightarrow (x, y) \in \theta^{J'}$,

$\Rightarrow x\theta^{J'} = y\theta^{J'}$,

$\Rightarrow a = b$. Thus $\hat{\hat{\theta}}$ is a monomorphism and so an isomorphism and our result is proved.

3. PRIME IDEALS AND Ψ^S

Next we discuss some relations between Ψ^S and prime ideals — the original motivation for looking at Ψ^S, and the source of applications of the congruence.

PROPOSITION 3.1. — *Let $\mathscr{L}$ be a lattice and let* S, T *be two* $\wedge$*-subsemilattices of $\mathscr{L}$ with* $S \subseteq T$. *Then the following are equivalent* :

(i) $\psi^{S,T} : \mathscr{L} // S \to \mathscr{L} // T$ *is an isomorphism.*

(ii) *For any* $t \in T$ *there is* $s \in S$ *such that* $t \wedge s \in S$.

(iii) *For any prime ideal of $\mathscr{L}$,* $P \cap T \neq \Box$ *implies* $P \cap S \neq \Box$.

Proof. Suppose that $\psi^{S,T}$ is bijective. Then for $a, b \in L // S$, $a\psi^{S,T} = b\psi^{S,T}$ implies $a = b$, i. e. if $a = x\psi^S$ and $b = y\psi^S$ then by (2.2) $x\psi^T = y\psi^T$ implies $x\psi^S = y\psi^S$. Thus, $\psi^{S,T}$ being bijective means that $\Psi^T \leqslant \Psi^S$, which, combined with $S \subseteq T$, implies that $\Psi^T = \Psi^S$. Hence any $t \in T$ must be congruent to some $s' \in S$ i. e. there is $s \in S$ with $t \wedge s = s' \wedge s \in S$. We have proved (i) $\Rightarrow$ (ii).

To prove (ii) $\Rightarrow$ (iii) we assume (ii) and take a prime ideal P such that $P \cap T \neq \Box$. For $t \in P \cap T$ we have $s \in S$ such that $t \wedge s \in S$. Clearly $t \wedge s \in P$ and so $t \wedge s \in P \cap S$ proving that $P \cap S \neq \Box$.

Finally, we assume (iii) and prove $\Psi^T = \Psi^S$. Let $(a, b) \in \Psi^T$ i.e. $a \wedge t = b \wedge t$ for some $t \in T$. Suppose $S \cap (t) = \Box$. Then there is a prime ideal P such that $t \in P$ and $P \cap S = \Box$. But this contradicts (iii) and so $S \cap (t) \neq \Box$. Then there is $s \leqslant t$, $s \in S$ and so $a \wedge s = a \wedge t \wedge s = b \wedge t \wedge s = b \wedge s$ and $(a, b) \in \Psi^S$. As noted above, $S \subseteq T$ implies $\Psi^S \leqslant \Psi^T$ and we have proved that $\Psi^S = \Psi^T$ which is equivalent to (i).

In the case when $S = L \backslash P$ is the set-complement of a prime ideal P of $\mathscr{L}$, we write (for brevity) $\mathscr{L}_P$ instead of $\mathscr{L} // (L \backslash P)$. Such lattices have a unique maximal ideal.

PROPOSITION 3.2. — *Let* $\mathscr{L}$ *be a lattice and* S *a* $\wedge$*-subsemilattice of* $\mathscr{L}$.

(i) *The map* $(\psi^{S})^{-1} : I(\mathscr{L} \,//\, S) \to I(\mathscr{L})$ *defines an order-isomorphism between* $\mathscr{P}_{\mathscr{L}//S}$ and $\{P \in \mathscr{P}_{\mathscr{L}} : P \cap S = \square\}$.

(ii) *If* $P' \in \mathscr{P}_{\mathscr{L}//S}$ and $P = P'(\psi^{S})^{-1}$ *then there is a canonical isomorphism* $\alpha : \mathscr{L}_{P} \overset{\simeq}{\to} (\mathscr{L} \,//\, S)_{P'}$.

Proof. (i) It is well known that the inverse image of a prime ideal under an epimorphism is again a prime ideal. Since $S \subseteq 1_{\mathscr{L}//S}(\psi^{S})^{-1}$ and $1_{\mathscr{L}//S} \notin P'$ where $P' \in \mathscr{P}_{\mathscr{L}//S}$, we deduce that $P = P'(\psi^{S})^{-1}$ satisfies $P \cap S = \square$. To show that $(\psi^{S})^{-1}$ is bijective (it is obviously order-preserving) when restricted to $\mathscr{P}_{\mathscr{L}//S}$, we will prove that if $P \cap S = \square$ for $P \in \mathscr{P}_{\mathscr{L}}$, then $P\psi^{S}(\psi^{S})^{-1} = P$. Clearly $P\psi^{S}(\psi^{S})^{-1} \supseteq P$. Suppose $x \in P\psi^{S}(\psi^{S})^{-1}$. Then $x\psi^{S} \in P\psi^{S}$ and so there is $p \in P$ and $s \in S$ with $x \wedge s = p \wedge s$. But $s \notin P$ and $p \wedge s \in P$ and so $x \in P$ proving the desired inclusion. Thus the assertion of (i) is now proved.

(ii) Consider the diagram, where $Q = L\backslash P$ and $Q' = (L \,//\, S)\backslash P'$. Define α in the obvious manner i. e. for $a = x\psi^{Q}$ where $x \in L$, we write $a\alpha = x\psi^{S}_{\mathscr{L}}\psi^{Q'}_{\mathscr{L}//S}$. A routine check shows that α is well-defined and thence, since the diagram commutes, an epimorphism. If $a\alpha = b\alpha$ for $a = x\psi^{Q}$ and $b = y\psi^{Q}$ we deduce that

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\ \psi^{S}\ } & \mathscr{L} \,//\, S \\ \Big\downarrow{\scriptstyle \psi^{Q}} & & \Big\downarrow{\scriptstyle \psi^{Q'}_{\mathscr{L}//S}} \\ \mathscr{L}_{P} & \xrightarrow{\ \alpha\ } & (\mathscr{L} \,//\, S)_{P'} \end{array}$$

$(x\psi^{S}, y\psi^{S}) \in \Psi^{Q'}$. Or, for some $q' \in Q'$

$x\psi^{S} \wedge q' = y\psi^{S} \wedge q'$. Now $q' = q\psi^{S}$ for $q \in Q = L\backslash P$ and so $(x \wedge q)\psi^{S} = (y \wedge q)\psi^{S}$ proving that there is $s \in S$ with $x \wedge q \wedge s = y \wedge q \wedge s$. But $1_{\mathscr{L}//S} \notin P'$ and so $P \cap S = \square$ which proves that $S \subseteq L\backslash P = Q$. Hence $q \wedge s \in Q$ and $(x, y) \in \Psi^{Q}$ and so $a = x\psi^{Q} = y\psi^{Q} = b$ and this injectivity implies that α is an isomorphism.

Recall that $h(P) = \{P_1 \in \mathscr{P}_{\mathscr{L}} : P_1 \supseteq P\}$.

PROPOSITION 3.3. — *Let* $\mathscr{L}$ *be a lattice and* P *a prime ideal of* $\mathscr{L}$. *For any proper ideal* $J' \in I(\mathscr{L}_{P})$ *the set* $J'(\psi^{L\backslash P})^{-1} = J \in I(\mathscr{L})$ and

(i) $\mathscr{L}_{P}/J' \overset{\simeq}{\to} (\mathscr{L}/J)_{P/J}$.

(ii) *the map* $J' \to J'(\psi^{L\backslash P})^{-1}$ *restricted to prime ideals of* $\mathscr{L}_{P}$ *is an order isomorphism between* $h(P)$ *and* $\mathscr{P}_{\mathscr{L}_{P}}$. *Further, if* $R' \in \mathscr{P}_{\mathscr{L}_{P}}$ *and* $R = R'(\psi^{L\backslash P})^{-1}$ *then* $\mathscr{L}_{R} \cong (\mathscr{L}_{P})_{R'}$.

Proof. Firstly (i) is an immediate consequence of (2.3) above with $S = L\backslash P$. Also (ii) follows from (3.2) (i) above which the final part is a case of (3.2) (ii).

4. THE CONGRUENCE R^{J} : GENERALITIES

The congruence which interests us in this section is defined for any ideal $J \in I(\mathscr{L})$ by

$$(a, b) \in R^{J} \underset{\mathrm{Df}}{\equiv} (a : J)^{*} = (b : J)^{*}.$$

Since this relation was proved to be a congruence in [4] we omit the routine proof of this fact. We write $D^J = \{a \in L : (a : J)^* = J\}$.

PROPOSITION 4.1. — *For* $\mathscr{L}$ *a lattice and* J *an ideal of* $\mathscr{L}$:

(i) R^J *is the largest congruence with* J *as a congruence class.*

(ii) D^J (*when non-empty !*) *is the unit congruence class for* R^J *and hence a dual ideal.*

(iii) $\mathscr{L}/R^J$ *is a disjunctive lattice with a unit when* $D^J \neq \square$.

Proof. (i) Let Φ be a congruence with J as a congruence class. Then $(x, y) \in \Phi$ implies that $(x \wedge t, y \wedge t) \in \Phi$ for any $t \in L$. Thus we must have $x \wedge t \in J$ iff $y \wedge t \in J$ and so $(x : J)^* = (y : J)^*$, giving $(x, y) \in R^J$. This proves that $\Phi \leqslant R^J$.

(ii) D^J is clearly a congruence class for R^J since $a \in D^J$ if and only if $x \wedge a \in J \Rightarrow x \in J$. Moreover $a \in D^J$ and $b \geqslant a$ imply $b \in D^J$. So D^J is the last congruence class for R^J and consequently a dual ideal.

(iii) We shall omit this since it follows from results in [4].

COROLLARY 4.2. — *If* $a \wedge d = b \wedge d$ *for* $d \in D^J$, *then* $(a, b) \in R^J$.

The canonical epimorphism associated with R^J is written ρ^J. There is also a dual construction giving a congruence $\check{R}^F$ which is the largest congruence having a given dual ideal F as a congruence class. We omit the details.

PROPOSITION 4.3. — *For any ideal* J *of* $\mathscr{L}$, *the following are equivalent* :

(i) $\mathscr{L}/R^J$ *is a Boolean lattice* ;

(ii) *For any* $x \in L$ *there is* $x' \in L : x \wedge x' \in J,\ x \vee x' \in D^J$;

(iii) *For any* $x \in L$ *there is* $x' \in L : ((x : J)^* : J)^* = (x' : J)^*$.

Proof. (i) $\Rightarrow$ (ii). If $\mathscr{L}/R^J$ is complemented then for any $x \in L$ there is $x' \in L$ such that $x\rho^J$ and $x'\rho^J$ are complements in $\mathscr{L}/R^J$ i. e. $x\rho^J \wedge x'\rho^J = 0_{\mathscr{L}/J}$ and $x\rho^J \vee x'\rho^J = 1_{\mathscr{L}/J}$. This means (by (4.1) above) that $x \wedge x' \in J$ and $x \vee x' \in D^J$.

(ii) $\Rightarrow$ (iii). Since $x \wedge x' \in J$ we already have $(x) \subseteq (x' : J)^*$ and so $((x : J)^* : J)^* \subseteq (((x' : J)^* : J)^* : J)^* = (x' : J)^*$ follows. For the reverse, suppose that $t \wedge x' \in J$ and that $u \in (x : J)^*$. Then $t \wedge u \wedge (x \vee x') \in J$ and so, since $x \vee x' \in D^J$ we deduce that $t \wedge u \in J$ proving that $t \in ((x : J)^* : J)^*$. Thus the equality is proved.

(iii) $\Rightarrow$ (i). The direct proof of this implication is longer and is omitted. It is much easier to prove (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) since the latter are the easy reverses of proofs above.

Our next proposition relates R^J to congruences with D^J as a specified congruence class.

PROPOSITION 4.4. — *Suppose that* $\mathscr{L}$ *satisfies one* (*and hence all*) *of the conditions of* (4.3). *Then* R^J *is the largest congruence with* D^J *as a congruence class.*

Proof. Let Φ be a congruence on $\mathscr{L}$ with D^J as a congruence class. Then if $(x, y) \in \Phi$ we have $(x \vee t, y \vee t) \in \Phi$ for any $t \in L$. This means that $x \vee t \in D^J$ iff $y \vee t \in D^J$ or, equivalently,

$$(x \vee t : J)^* = J \text{ iff } (y \vee t : J)^* = J.$$

Rewriting this as

$$(x : \mathrm{J})^* \cap (t : \mathrm{J})^* = \mathrm{J} \text{ iff } (y : \mathrm{J})^* \cap (t : \mathrm{J})^* = \mathrm{J}$$

we use (iii) of 4.3 above to obtain :

$$((x' : \mathrm{J})^* : \mathrm{J})^* \cap ((t' : \mathrm{J})^* : \mathrm{J})^* = \mathrm{J} \text{ iff } ((y' : \mathrm{J})^* : \mathrm{J})^* \cap ((t' : \mathrm{J})^* : \mathrm{J})^* = \mathrm{J}$$

i. e. $((x' \wedge t' : \mathrm{J})^* : \mathrm{J})^* = \mathrm{J}$ iff $((y' \wedge t' : \mathrm{J})^* : \mathrm{J})^* = \mathrm{J}$

Now this means $t' \wedge x' \in \mathrm{J}$ iff $t' \wedge y' \in \mathrm{J}$ and since any $s \in \mathrm{L}$ is of the form t' we have proved that

$$(x' : \mathrm{J})^* = (y' : \mathrm{J})^*$$

which implies that

$$(x : \mathrm{J})^* = ((x' : \mathrm{J})^* : \mathrm{J})^* = ((y' : \mathrm{J})^* : \mathrm{J})^* = (y : \mathrm{J})^*$$

and so

$$(x, y) \in \mathrm{R}^{\mathrm{J}}.$$

Thus $\Phi \leqslant \mathrm{R}^{\mathrm{J}}$.

The final result of this section tells us that we might as well continue to the next paragraph.

Proposition 4.5. — *Let* J *be any ideal of the lattice* $\mathscr{L}$. *Then the map* $\varphi : \mathrm{L}/\mathrm{R}^{\mathrm{J}} \to (\mathrm{L}/\mathrm{J})/\mathrm{R}$ *given by* $(x\rho^{\mathrm{J}})\varphi = x\theta^{\mathrm{J}}\rho_{\mathscr{L}/\mathrm{J}}$ *defines an isomorphism*

$$\mathscr{L}/\mathrm{R}^{\mathrm{J}} \overset{\cong}{\to} (\mathscr{L}/\mathrm{J})/\mathrm{R}$$

Proof. We have the diagram

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\theta^{\mathrm{J}}} & \mathscr{L}/\mathrm{J} \\ \rho^{\mathrm{J}} \downarrow & & \downarrow \rho_{\mathscr{L}/\mathrm{J}} \\ \mathscr{L}/\mathrm{R}^{\mathrm{J}} & \xrightarrow{\varphi} & (\mathscr{L}/\mathrm{J})/\mathrm{R} \end{array}$$

We prove φ is injective. If $a_1\varphi = a_2\varphi$ for $a_i \in \mathrm{L}/\mathrm{R}^{\mathrm{J}}$ $i = 1, 2$. Then $a_i = x_i\rho^{\mathrm{J}}$ for $x_i \in \mathrm{L}$ $i = 1, 2$, and

$$x_1\theta^{\mathrm{J}}\rho_{\mathscr{L}/\mathrm{J}} = x_2\theta^{\mathrm{J}}\rho_{\mathscr{L}/\mathrm{J}}$$

This means that $(x_1\theta^{\mathrm{J}}, x_2\theta\mathrm{J}) \in \mathrm{R}_{\mathscr{L}/\mathrm{J}}$ or,

$$x_1\theta^{\mathrm{J}} \wedge t = 0_{\mathscr{L}/\mathrm{J}} \text{ iff } x_2\theta^{\mathrm{J}} \wedge t = 0_{\mathscr{L}/\mathrm{J}} \text{ for } t \in \mathrm{L}/\mathrm{J}.$$

Now $t = s\theta^{\mathrm{J}}$ for some $s \in \mathrm{L}$ and so

$$x_1\theta^{\mathrm{J}} \wedge s\theta^{\mathrm{J}} = 0_{\mathscr{L}/\mathrm{J}} \text{ iff } x_2\theta^{\mathrm{J}} \wedge s\theta^{\mathrm{J}} = 0_{\mathscr{L}/\mathrm{J}} \text{ for any } s \in \mathrm{L}.$$

i. e. $(x_1 \wedge s)\theta^{\mathrm{J}} = 0_{\mathscr{L}/\mathrm{J}}$ iff $(x_2 \wedge s)\theta^{\mathrm{J}} = 0_{\mathscr{L}/\mathrm{J}}$ for any $s \in \mathrm{L}$

or

$$x_1 \wedge s \in \mathrm{J} \text{ iff } x_2 \wedge s \in \mathrm{J} \text{ for any } s \in \mathrm{L}.$$

Thus $(x_1, x_2) \in \mathrm{R}^{\mathrm{J}}$ and so $a_1 = x_1\rho^{\mathrm{J}} = x_2\rho^{\mathrm{J}} = a_2$ and φ is injective.

φ is clearly an epimorphism and so our result is proved.

5. THE CONGRUENCE R

Throughout this paragraph $R = R^{(0)}$ is the congruence defined by $(x, y) \in R$ iff $(x)^* = (y)^*$. The dual ideal $D^{(0)}$ is written D and $\rho^{(0)}$ is written ρ. In some recent works we have presented results concerning R which we now recall in the present context; some extensions are then noted.

PROPOSITION 5.1. — *The map* $\alpha : L/R \to L^{**} = \{(x)^{**} : x \in L\}$ *given by* $(x\rho)\alpha = (x)^{**}$ *defines an isomorphism* $\mathscr{L}/R \xrightarrow{\cong} \mathscr{L}^{**}$,

Proof. This is a special case of Lemma 2.3 of [5].

Now the results concerning R^J can be strengthened for $J = (0)$.

PROPOSITION 5.2. — $\mathscr{L}/R$ *is an m-complete Boolean lattice iff for any* $A \subseteq L$ *with* $|A| \leqslant m$, *there is* $a' \in L : A^{**} = (a')^*$.

Proof. This is a special case of Theorem 2 of [5].

COROLLARY 5.3. — *If* $\mathscr{L}$ *is a distributive pseudo-complemented lattice that is closed under the formation of m-ary joins, and that satisfies* $x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} x \wedge y_i$ *when* $|I| \leqslant m$, *then* $\mathscr{L}/R$ $(\cong \mathscr{L}^{**})$ *is an m-complete Boolean lattice.*

Proof. We must show that $\mathscr{L}$ satisfies the condition of 5.2. Take $A \subseteq L$ with $|A| \leqslant m$ and let $a = \bigvee \{x : x \in A\}$. It is immediate that $(a^*) = (a)^* = A^*$, and so $A^{**} = (a^*)^*$ proving our result.

Remark. This Corollary tidies up G. Birkhoff's treatment ([1] p. 130) of Glivenko's Theorem. In that treatment, the fact that $\mathscr{L}$ is Brouwerian (rather than just pseudo-complemented) is not used, nor is the degree of completeness of $\mathscr{L}^{**}$ (C in the book) clarified. A related result is :

COROLLARY 5.4. — If $\mathscr{L}$ is pseudo-complemented, then the normal completion of the Boolean lattice $\mathscr{L}/R$ is the complete Boolean lattice $\mathscr{I}_{\mathscr{L}}/R$.

Proof. This proved exactly as in Theorem 3 of [5].

A sharpening of (4.4) is given in the following result from [6] :

PROPOSITION 5.4. — *Suppose that* $\mathscr{L}/R$ *is Boolean. Then* $\check{\theta}^D = R = \check{R}^D$, *i. e.* R *is the unique congruence with* D *as a congruence class.*

We close this section by giving some results concerning the case when $\mathscr{L}/R$ is Boolean, which are not included in [6]. Some of these refer to the spaces of prime ideals and minimal prime ideals resp. of $\mathscr{L}$, each equipped with the hull-kernel topology. Rather than give full details of these spaces, we refer to the articles [7] and [8] where full discussions are given. One result worth noting is that for any distributive lattice $\mathscr{L}$, $\mathscr{L}/R$ and $\mathscr{L}$ have canonically homeomorphic spaces of minimal prime ideals; this is proved in the paper [3] by J. Kist.

PROPOSITION 5.5. — *Let* $\mathscr{L}$ *be a distributive lattice with zero. Then the following are equivalent :*

(i) $\mathscr{L}/R$ *is Boolean and* D *is principal, i. e.* $D = [d]$;

(ii) *The set* $\mathcal{M}_{\mathcal{L}}$ *of all minimal prime ideals is a compact open subset of Spec* $(\mathcal{L})$, *the space of prime ideals of* $\mathcal{L}$ *with the hull-kernel topology.*

Proof. (i) implies (ii). Assume that $\mathcal{L}/R$ is Boolean, and that $D = [d]$. Then for any prime ideal P of $\mathcal{L}$, if $d \notin P$ then P is minimal by Theorem 1 of [6]. Conversely, if $P \in \mathcal{M}_{\mathcal{L}}$ then $d \notin P$ since minimal prime ideals do not contain dense elements — see [3]. This means that $\mathcal{M} = \mathcal{P}(d) = \{P \in \mathcal{P}_{\mathcal{L}} : d \notin P\}$ and so $\mathcal{M}_{\mathcal{L}}$ is a compact-open subset of Spec $(\mathcal{L})$ — see [8].

(ii) implies (i). If $\mathcal{M}_{\mathcal{L}}$ is a compact open subset of Spec $(\mathcal{L})$, then there is $d \in L$ with $\mathcal{M}_{\mathcal{L}} = \mathcal{P}(d)$. By a known result on minimal prime ideals (see [3]) d must satisfy $(d)^* = (0)$ since d belongs to no minimal prime ideal. Thus $d \in D$ and if a prime ideal P contains a dense element less than d and does not contain d, P must be minimal which is a contradiction. Hence d is the least dense element and so $D = [d]$. Finally the reverse of Theorem 1 of [6] proves that $\mathcal{L}/R$ is Boolean. Our result is proved.

We now consider minimum elements of R-classes and relate them to the foregoing. Write $\overline{D} = \cap \{(d) : d \in D\}$.

PROPOSITION 5.6. — *Suppose that* $\mathcal{L}/R$ *is Boolean. Then* $\overline{D}$ *is the set of minimum elements of* R*-classes of* $\mathcal{L}$.

Proof. If x is the minimum of $x\rho\rho^{-1}$ (the R-class containing x) then $x \in \overline{D}$. For if $x \notin \overline{D}$ there is $d \in D$ with $x \wedge d < x$. But $(x, x \wedge d) \in R$ contradicting x being the minimum element.

Conversely, if $x \in \overline{D}$ then x is the minimum element of $x\rho\rho^{-1}$. For if $(x, y) \in R$ there is (by (5.4) above) $d \in D$ such that $x \wedge d = y \wedge d$. But $x \wedge d = x$ since $x \in \overline{D}$ and so $x = y \wedge d$ which proves that $x \leqslant y$.

COROLLARY 5.7. — *No two distinct-elements of* $\overline{D}$ *are* R*-equivalent.*

Our final result determines when R-classes possess minimum elements.

PROPOSITION 5.8. — *Suppose that* $\mathcal{L}/R$ *is Boolean. Then the following are equivalent :*

(i) $\mathcal{M}_{\mathcal{L}}$ *is a compact-open subset of Spec* $(\mathcal{L})$;

(ii) $D = [d]$, *i. e. the dense dual ideal of* $\mathcal{L}$ *is principal* ;

(iii) *Every* R*-class possesses a minimum element.*

Proof. (i) and (ii) are already shown to be equivalent.

(ii) implies (iii). For any R-class $x\rho\rho^{-1}$ we assert that $x \wedge d$ is the minimum element of $x\rho\rho^{-1}$. For, if $y \in x\rho\rho^{-1}$ then $x \wedge d = y \wedge d$ and so $y \geqslant x \wedge d$.

(iii) implies (i). If every R-class has a minimum element then D must also, since D is an R-class, i. e. $D = [d]$ for some $d \in L$.

COROLLARY 5.9. — *Under any of the equivalent conditions of* 5.8, *the map* $\alpha : L/R \to \overline{D}$ *given by* $(x\rho)\alpha = x \wedge d$ *defines an isomorphism*

$$\mathcal{L}/R \cong \overline{D} = (d).$$

Remark. These last few results are closely related to, and were stimulated by, work of J. Varlet [9].

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