Bulletin de la Société Royale des Sciences de Liège, 38e année, n° 3-4, 1969, pp. 73-85

ON A PRINCIPLE OF « MINIMUM REYNOLDS NUMBER » AND THE MALKUS THEORY OF TURBULENT CHANNEL FLOW

by JACQUES C. J. NIHOUL
Institut de Mathématique
Université de Liège, Belgium

SUMMARY

The paper reexamines some aspects of channel flow stability and turbulence. The approach followed draws its inspiration from the Liapounov stability concepts rather than the classical Orr-Sommerfeld perturbation technique and originates from Serrin's variational theorem of stability. It leads to a principle of « minimum Reynolds number » with interesting applications to the study of stability, equilibrium flows and stationary turbulence, throwing some new light on the Malkus theory.

1. INTRODUCTION

There is obviously a close relationship between the problems of stability and turbulence, if only because turbulence arises in most cases from the breaking of stability. The non-linear interactions of finite amplitude disturbances provide, to some extent, a representation of what turbulence — the degree of complexity increasing — may be ultimately. On the other hand, the existence, under given physical conditions, of a turbulent flow with definite average characteristics suggests that the realization of the observed turbulent state is also related to some stability criteria.

This idea was exploited by Malkus in his well-known theory of turbulent shear flow (Malkus 1956). In this paper and in a later paper by Nihoul on MHD turbulent channel flow (Nihoul 1966), essential properties of the flow such as the mean velocity profile or the mean friction coefficient were sought with the help of an equivalent stability problem defined on the basis of a series of postulates propounded by Malkus.

Malkus's hypotheses may be formulated as follows :

1° The mean flow is stable to infinitesimal disturbances. Thus the mean velocity profile cannot have an inflexion point and the mean vorticity gradient has everywhere the same sign. Under definite physical conditions, the statistically steady turbulent flow which is observed is that particular flow for which the critical Reynolds number of the mean flow stability is the corresponding physical Reynolds number.

2° The turbulent velocity fluctuations have a negligible — actually slightly stabilizing — *direct* influence on the evolution of a perturbation. They have a determinant *indirect* influence on the stability problem through the modification of the mean velocity profile under the action of the average turbulent Reynolds stresses.

3° The average Reynolds stresses and mean velocity spectra have a smallest

Manuscrit reçu le 20 mars 1969.

scale of motion which is related to the smallest marginally stable motion according to the stability theory.

4° Within the constraints which derive from the Navier-Stokes equations, the boundary conditions and the assertions 1, 2 and 3, the dissipation rate is maximum for fixed mean flow.

Despite indisputable successes of the Malkus theory, the basic postulates have been much debated and although many arguments can be found to ascertain their validity, from a physical point of view (Malkus 1961, Nihoul 1967*a*), there seems to remain doubt on their proper interpretation (Reynolds and Tiederman 1967).

One of the difficulties in discussing the Malkus theory is that, being closely related to the stability problem of channel flows, its concrete formulation and application by Malkus resort essentially to the theory of the Orr-Sommerfeld equation. As pointed out by Nihoul (Nihoul 1968), it is not quite clear then, in a critical discussion such as the one by Reynolds (Reynolds and Tiederman 1967), which argument actually questions the fundamental principles of the theory and which aims in fact at the — sofar only — approximate solution of the Orr-Sommerfeld problem.

For that reason, we would like to reexamine here some aspects of the channel flow stability and turbulence problems following a different approach. This approach which draws its inspiration from Liapounov's stability method rather than the classical Orr-Sommerfeld perturbation technique originates from Serrin's variational theorem of stability (Serrin 1959) and leads to a principle of « minimum Reynolds number » with interesting applications to the analysis of stability, equilibrium flows and stationary turbulence, throwing some new light on the Malkus theory (*).

2. STABILITY OF LAMINAR FLOW

To begin with, we recall and briefly comment a theorem demonstrated by Serrin (Serrin 1959).

We consider a basic fluid motion occupying a bounded region Ω of space. We assume that the velocity field of the basic flow is altered at some initial instant t_0. Following Serrin, we say that the basic motion is stable (or, more precisely, « stable in the mean ») if the kinetic energy E of the perturbation motion tends to zero as t tends to infinity i. e. if the rate of change $\frac{dE}{dt}$ is negative. If $\frac{dE}{dt}$ is zero, we say that the basic motion is marginally stable. Restricting attention to an incompressible fluid contained by rigid walls, we obtain readily, in non-dimensional form, (Reynolds 1895, Orr 1907, Serrin 1959)

$$(1) \qquad \frac{dE}{dt} = -R \int_\Omega [\mathbf{w} . D . \mathbf{w}]\, d\Omega - \int_\Omega || \nabla \times \mathbf{w} ||^2 \, d\Omega = RI_1 - I_2$$

where D is the deformation tensor of the basic motion $\left[\text{i. e. } D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$, $\mathbf{w}$ is the velocity of the perturbation motion and R is the Reynolds number, $R = \frac{\bar{u}L}{\nu}$;

(*) There is also some analogy and some bearing on the ideas put forward by Lumley (1965).

ν is the kinematic viscosity, $\bar{u}$ and L respectively the reference velocity and the reference length which were used to make D, **w** and $\nabla \times \mathbf{w}$ non-dimensional.

On the right-hand side of (1), the term RI_1 represents the rate of energy transfer from the mean flow to the disturbance, the term I_2 is always positive and represents the rate of dissipation of the disturbance energy. The solenoïdal vector field **w** which maximizes I_1 for constant I_2 (or minimizes I_2 for constant I_1) must satisfy the continuity equation

$$\nabla . \mathbf{w} = 0 \tag{2}$$

and the Euler-Lagrange equations

$$\nabla^2 \mathbf{w} = \nabla \lambda + \mu \mathbf{w} . \mathrm{D} \tag{3}$$

where λ and μ are Lagrange multipliers. Eqs (2) and (3), with the conditions that **w** be zero on the boundary, set an eigenvalue problem for the constant multiplier μ.

Serrin demonstrated the following theorem (Serrin 1959) : The eigenvalues μ constitute a bounded set. If $\tilde{\mathbf{w}}$ is the maximizing vector associated with the smallest eigenvalue $\tilde{\mu}$, then

$$\tilde{\mu} = - \frac{\int_\Omega \| \nabla \times \tilde{\mathbf{w}} \|^2 d\Omega}{\int_\Omega \tilde{\mathbf{w}} . \mathrm{D} . \tilde{\mathbf{w}} \, d\Omega} \tag{4}$$

The basic motion is stable if $R < \tilde{\mu}$.

This theorem holds when Ω is unbounded provided the flow geometry is such that the disturbances can be assumed spatially periodic at each instant, as for instance in the case of Poiseuille flow. The region Ω can then be chosen to cover exactly one wave length and the boundary integrals at either end of Ω, neither of which vanishes separately, just cancel one another. Formula (1) may therefore be assumed to hold in this situation.

Serrin's definition of « stability in the mean » which he actually borrowed from the early works of Orr (1907) meets the stability concepts of Liapounov (Liapounov 1947) which were recently generalized to continuum mechanics by Zubov, Movchan, Knops et al (Zubov 1964, Movchan 1959, 1960, 1963, Knops and Wilkes 1966, Gilbert and Knops 1967). In the functional space of the field variables, the perturbation energy E is indeed a suitable metric ρ which measures the distance from the laminar motion to the perturbed motion. Serrin's theorem ensures then that the distance $\rho \equiv E$ decreases with time from its initial value along any natural trajectory in the functional space, for any $R < \tilde{\mu}$ and any perturbation (since this is so for the worst possible one $\tilde{\mathbf{w}}$) (*).

The Euler-Lagrange equations (3) are linear in **w** and — although there is no restriction in Serrin's theory on the magnitude of the perturbation — finite amplitude solutions of (3) will not normally be solutions of the Navier-Stokes equations. For instance, in the case of Couette flow between rotating cylinders treated in illustration by Serrin (Serrin 1959), eq (3) admits solutions of the form

$$w_r = \hat{w}_r(r) \cos kz, \; w_\theta = \hat{w}_\theta(r) \cos kz, \; w_z = \hat{w}_z(r) \sin kz \tag{5}$$

(*) The Orr-Serrin criterion is actually a little stronger than Liapounov's as the latter — especially in the context of the second method of the Liapounov functional — only requires that it be possible to confine the perturbed motion in *any* domain $\rho < B$ by restricting the initial disturbance to *some* region $\rho < b$ where b can be smaller than B.

which cannot satisfy the Navier-Stokes equations. But then, solutions of this type have been observed experimentally to describe satisfactorily the secondary motion occuring in Couette flow and it is customary to postulate disturbances of this form in the non-linear analysis of stability. One may refer here to the work of Stuart (e. g. Stuart 1958) and more recently to Pritchard's application of Liapounov's second method to the study of the Benard and Couette problems stability (Pritchard 1968).

The merit of Serrin's approach seems precisely that it provides us with a well defined « equivalent » linear problem which arises quite naturally from the Liapounov Movchan stability theory ; which, among others, admit solutions of the classical tentative forms and which is consolidated by a variational theorem which ascertains the quality of the approximation. (The results obtained by Serrin for the stability of laminar Couette flow were in excellent agreement with the experiments.)

The condensation of the theory in a variational principle has the further advantage of allowing direct methods of analysis and facilitating eventual numerical calculations. This principle may be formulated as a principle of minimum Reynolds number

$$\mathscr{R} = \frac{\int_{\Omega} \| \nabla \times \mathbf{w} \|^2 \, d\Omega}{- \int_{\Omega} \mathbf{w} \cdot \mathrm{D} \cdot \mathbf{w} \, d\Omega} \tag{6}$$

for marginal stability (see for instance eq. 4).

3. EQUILIBRIUM FLOWS

In the papers by Serrin and Pritchard (Serrin 1959, Pritchard 1968), the perturbation is the difference between the perturbed motion and the original laminar one. (Hence, the Euler-Lagrange equations (3) are linear, the deformation tensor D referring to the known laminar motion).

In view of the spatial periodicity of the perturbation, however, — and with statistically steady turbulence in mind for a later study — it is convenient to take space averages (with respect to one spatial dimension, at least) and to separate the flow into a mean part and a disturbance part where the latter has zero mean.

An equilibrium state in which the rate of transfer of energy from the mean flow to the disturbance balances precisely the rate of viscous dissipation of energy of the disturbance is called an equilibrium flow (Stuart 1958). In an equilibrium flow, the disturbance has a definite finite amplitude and the mean flow — distorted from its original laminar form — is steady.

In certain problems, such as the circular Couette flow, several states of equilibrium flow can be observed. In other problems like the flow between parallel walls, the transition from the laminar regime to turbulence seems on the other hand to occur suddenly.

It is readily seen that Stuart's definition of the disturbance (i. e. departure from the mean) does not invalidate the expression (1) of the rate of change of the perturbation energy. Indeed writing the Navier-Stokes equations for $\mathbf{v} = \mathbf{n} + \mathbf{w}$ where $\mathbf{u}$ denotes the mean velocity and $\mathbf{w}$ the disturbance, taking the mean and substracting, we get (in non dimensional form)

(7) $$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}[u_i u_j + \langle w_i w_j \rangle] = -\frac{\partial p}{\partial x_i} + \mathrm{R}^{-1}\nabla^2 u_i$$

(8) $$\frac{\partial w_i}{\partial t} + \frac{\partial}{\partial x_j}[w_j u_i + w_i u_j + w_i w_j - \langle w_i w_j \rangle] = -\frac{\partial \pi}{\partial x_i} + \mathrm{R}^{-1}\nabla^2 w_i$$

where $\langle\ \rangle$ indicates an average and where p and π are the non-dimensional mean pressure and perturbation pressure.

Multiplying (8) by w_i, summing over i and integrating over Ω (the volume Ω covers one wave-length in the directions of periodicity and is limited to solid walls otherwise), we obtain

(9) $$\frac{d\mathrm{E}}{dt} = -\mathrm{R}\int_\Omega w_i w_j \frac{\partial u_i}{\partial x_j}\, d\Omega + \int_\Omega w_i \nabla^2 w_i \, d\Omega$$

$$= -\mathrm{R}\int_\Omega \mathbf{w}\,.\,\mathrm{D}\,.\,\mathbf{w}\, d\Omega - \int_\Omega \|\nabla \times \mathbf{w}\|^2 \, d\Omega$$

and comparing the two integrals in the right-hand side, we may define a « Reynolds number »

(10) $$\mathscr{R} = -\frac{\displaystyle\int_\Omega \|\nabla \times \mathbf{w}\|^2 \, d\Omega}{\displaystyle\int_\Omega \mathbf{w}\,.\,\mathrm{D}\,.\,\mathbf{w}\, d\Omega}$$

which is equal to the actual Reynolds number of the flow if the disturbance is stationary. In writing (9) and (10) we have taken into account that $\mathbf{w}$ has a zero mean and that the surface integrals vanish. The deformation tensor D refers now to the mean velocity $\mathbf{u}$ which must be counted among the unknowns of the problem.

Let us assume that, on this additional variable, we impose certain contraints (obtained from the boundary conditions or other physical considerations which we shall examine more specifically later) in such a way that the Euler-Lagrange equations associated with the minimization of $\mathscr{R}$ yield, in addition to eqs (3), the stationary form of (7) for the mean velocity profile. The eigenvalues of the boundary value problem set by these equations are the minima of $\mathscr{R}$ (cf. eq. 4) (*) and whenever the Reynolds number R is equal to one of these eigenvalues, the disturbance energy E is stationary for the corresponding minimizing vector which also satisfies the stationary form of (7) and could describe an equilibrium flow (**).

As a specific example, consider the flow of an incompressible newtonian fluid between two parallel planes at $x_2 = \pm 1$; the x_1 and x_2 axes being drawn in the direction of flow and normal to the planes, respectively.

We denote by $\langle\ \rangle$ space averages over x_1 and x_3 and put $\mathbf{v} = \mathbf{u} + \mathbf{w}$ where $\mathbf{u} = \langle \mathbf{v} \rangle = [u(x_2), 0, 0]$ is the mean velocity and $\mathbf{w}$ the disturbance of zero mean.

It is readily seen that the mean pressure gradient is a constant over the channel

(*) This result follows immediately from the form of the Euler-Lagrange equations (3) and could be exploited to ascertain the mathematical conditions under which there exist eigenvalues of the *non-linear* problem (Nehari 1960, 1961).

(**) With the same accepted approximation as in section 2 and in the same sense as in Stuart's work (e. g. Stuart 1958).

span. The pressure gradient is externally applied and the value of this constant is actually a boundary condition. Let

$$\text{(11)} \qquad \mathrm{R}\,\frac{\partial}{\partial x_1}\left\langle\frac{p}{\rho\bar{u}^2}\right\rangle = -\,\mathrm{Z}.$$

We shall assume that Z has the same value whether we consider laminar, equilibrium or turbulent flows.

The equation for the mean velocity can then be written, in the steady case,

$$\text{(12)} \qquad \frac{d^2u}{dx_2^2} = -\,\mathrm{Z} + \mathrm{R}\,\frac{d}{dx_2}\langle w_1 w_2\rangle.$$

In this equation, the mean velocity u and the fluctuationg velocity $\mathbf{w}$ are non dimensional. The reference velocity $\bar{u}$ is the bulk average velocity i. e.

$$\text{(13)} \qquad \int_{-1}^{1} u\,dx_2 = 2.$$

The average rate of work of the pressure gradient is given by

$$\text{(14)} \qquad \int_{-1}^{1} \mathrm{Z}u\,dx_2 = -\int_{-1}^{1} u\,\frac{d^2u}{dx_2^2}\,dx_2 + \mathrm{R}\int_{-1}^{1} u\,\frac{d}{dx_2}\langle w_1 w_2\rangle$$

i. e., integrating by parts and taking into account that $u = 0$ at $x_2 = \pm 1$

$$\text{(15)} \qquad 2\mathrm{Z} = \int_{-1}^{1} \dot{u}^2\,dx_2 - \mathrm{R}\int_{-1}^{1} \dot{u}\,\langle w_1 w_2\rangle\,dx_2$$

(Henceforth a dot denotes a derivation with respect to x_2).

The average rate of change of the disturbance energy is :

$$\text{(16)} \qquad \frac{d\mathrm{E}}{dt} = -\,\mathrm{R}\int_{-1}^{1} \langle w_1 w_2\rangle\,\dot{u}\,dx_2 - \int_{-1}^{1} \langle \|\nabla\times\mathbf{w}\|^2\rangle\,dx_2.$$

We may thus define

$$\text{(17)} \qquad \mathscr{R} = \frac{\displaystyle\int_{-1}^{1} \langle \|\nabla\times\mathbf{w}\|^2\rangle\,dx_2}{\displaystyle -\int_{-1}^{1} \langle w_1 w_2\rangle\,\dot{u}\,dx_2}.$$

If $\mathrm{R} = \mathscr{R}$ the energy of the disturbance is stationary and the global power balance may be written (combining 15 and 16)

$$\text{(18)} \qquad 2\mathrm{Z} = \int_{-1}^{1} \dot{u}^2\,dx_2 + \int_{-1}^{1} \langle \|\nabla\times\mathbf{w}\|^2\rangle\,dx_2$$

expressing that the energy supplied by the pressure forces is ultimately dissipated by viscosity.

We now seek the state vector $\tilde{u}$, $\tilde{\mathbf{w}}$ which minimizes $\mathscr{R}$ subject to conditions (2), (13), (18) and the boundary conditions

$$\text{(19)} \qquad \begin{array}{c} u = w_1 = w_2 = w_3 = 0 \\ \dot{u} = \mp\,\mathrm{Z} \end{array} \qquad \text{at } x_2 = \pm 1.$$

It is easy to see that the Euler-Lagrange equations of this problem yield eq (12) in addition to the equations (3) for **w**. Let, for instance, **w** be described by the following expressions (Meksyn 1964)

(20) $$w_1 = [a_1 \cos\theta + b_1 \sin\theta] \cos\gamma x_3$$

(21) $$w_2 = [a_2 \cos\theta + b_2 \sin\theta] \cos\gamma x_3$$

(22) $$w_3 = [a_3 \cos\theta + b_3 \sin\theta] \sin\gamma x_3$$

where $\theta = \alpha x_1 - \beta t$ and where the a's and b's are functions of x_2 and are related by the conditions of incompressibility i. e.

(23) $$\dot{a}_2 = -\alpha b_1 - \gamma a_3$$

(24) $$\dot{b}_2 = \alpha a_1 - \gamma b_3.$$

Introducing Lagrange multipliers λ_1, λ_2, μ_1, μ_2, μ_3 (the λ's are functions of x_2) the variational problem may be expressed as follows :

(25) $$\begin{aligned}\delta \int_{-1}^{1} \{(\dot{a}_1 - \alpha \dot{b}_2)^2 + (b_1 + \alpha a_2)^2 + (\dot{a}_3 + \gamma a_2)^2 + (\dot{b}_3 + \gamma b_2)^2 \\ + (\gamma a_1 + \alpha b_3)^2 + (\gamma b_1 - \alpha a_3)^2 + \mu_1 \ddot{u}(a_1 a_2 + b_1 b_2) \\ + 2\lambda_1(\dot{a}_2 + \alpha b_1 + \gamma a_3) + 2\lambda_2(\dot{b}_2 - \alpha a_1 + \gamma b_3) \\ + \mu_2 \ddot{u}^2 + \mu_3 u\}\, dx_2 = 0\end{aligned}$$

with the side conditions (13), (18), (23) and (24). The Euler-Lagrange equations may be written (after some manipulations) :

(26) $$\mathscr{D}^2 a_1 = \frac{1}{2} \mu_1 a_2 \ddot{u} - \alpha \lambda_2$$

(27) $$\mathscr{D}^2 b_1 = \frac{1}{2} \mu_1 b_2 \ddot{u} + \alpha \lambda_1$$

(28) $$\mathscr{D}^2 a_2 = \frac{1}{2} \mu_1 a_1 \ddot{u} - \dot{\lambda}_1$$

(29) $$\mathscr{D}^2 b_2 = \frac{1}{2} \mu_1 b_1 \ddot{u} - \dot{\lambda}_2$$

(30) $$\mathscr{D}^2 a_3 = \gamma \lambda_1$$

(31) $$\mathscr{D}^2 b_3 = \gamma \lambda_2$$

where

(32) $$\mathscr{D}^2 = \frac{d^2}{dx_2^2} - \alpha^2 - \gamma^2$$

(33) $$\frac{d}{dx_2} [\mu_1(a_1 a_2 + b_1 b_2) + 2\mu_2 \ddot{u}] = \mu_3.$$

Eqs (26) to (31) are the Fourier decomposition of eqs (3), with

(34) $$\lambda = (\lambda_1 \cos\theta + \lambda_2 \sin\theta) \cos\gamma x_3.$$

Their eigenvalues $\tilde{\mu}_1$ are the corresponding minima $\tilde{\mathscr{R}}$ of $\mathscr{R}$. With the help of (18) and (19), eq. (33) may then be written

$$\tilde{\mathscr{R}} \langle \tilde{w}_1 \tilde{w}_2 \rangle - Zx_2 = \dot{\tilde{u}}. \tag{35}$$

If $R = \mathscr{R}$, the disturbance energy is stationary for the eigensolution $\tilde{\mathbf{v}} = \tilde{\mathbf{u}} + \tilde{\mathbf{w}}$ and the corresponding mean flow $\tilde{u}$ satisfies then the steady state equation (12) describing an equilibrium flow.

We have sofar left aside the question of the existence and the determination of the eigenvalues of the non-linear problem (*). It is reasonable to expect, for each α and γ, a series $0 < \tilde{\mu}_1^1 < \tilde{\mu}_1^2 < \dots \tilde{\mu}_1^n$ of eigenvalues corresponding to eigensolutions of larger and larger transverse wave number (n, say) and it is tempting to speculate that the surfaces $\tilde{\mu}_1^n$ (α, γ) are nested into one another in a way similar to the sketch proposed by Spiegel by analogy with thermal turbulence (Spiegel 1962).

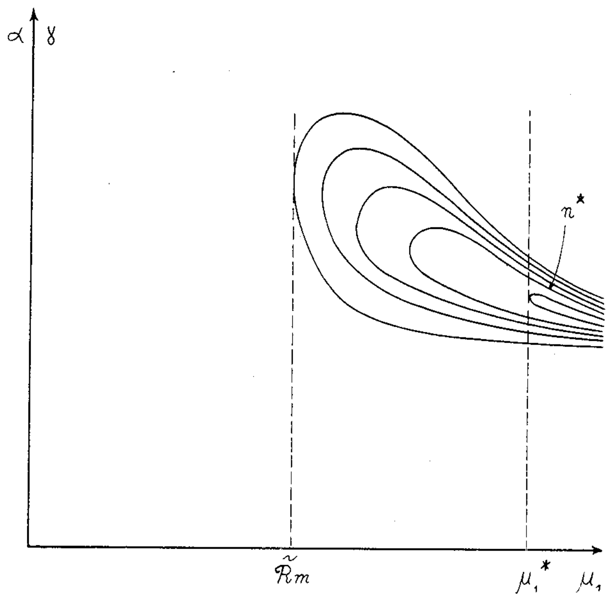

Fig. 1. — Tentative sketch of the intersections of the characteristic surfaces $\mu_1^n(\alpha, \gamma)$ with a plane α = constant or γ = constant.

Fig. 1 shows the tentative form of the intersections of these surfaces with a plane γ = constant or α = constant. If R is equal to the smallest eigenvalue $\tilde{\mathscr{R}}_m$, an equilibrium flow is conceivable characterized by unique values of the wave

(*) There is very little literature on the eigenvalues of non-linear boundary problems. — Certain methods of investigation such as the methods of Liapounov and Schmidt are becoming available (Vainberg and Trenogin 1962) and some works have been done recently in the domain (e. g. Kirchgässner 1960, 1961, Görtler et al. 1965). We have also made several interesting observations on the subject in connection with the variational study of secondary Couette flows which will be reported in a forthcoming publication.

numbers α and γ, hence displaying simple well identified periodicities and comparable to the first secondary flow observed in the circular Couette problem.

Furthermore, we may argue that, since the Euler Lagrange equations for **w** have the same form as in section 2 (*), the mean flow $\tilde{u}$ is then marginally stable to infinitesimal disturbances negligibly affecting (and affected by) the $\tilde{\mathbf{w}}$-field ; i. e., if $\tilde{u}$ were some laminar flow realized, as it is, by some appropriate device, without any superimposed disturbance, this laminar flow would be marginally stable (in the sense of Serrin) for the actual Reynolds number $\tilde{\mathscr{R}}_m$.

The realization of such an equilibrium flow depends however obviously on the possibility of finding a smallest eigenvalue $\tilde{\mathscr{R}}_m$ of the non-linear problem *beyond* the critical Reynolds number R_c of stability of the laminar motion to infinitesimal disturbances.

In the case of parallel flow, most investigations seem to indicate that the modification of the laminar profile by a superimposed finite disturbance is a distabilizing effect (e. g. Stuart 1962, Reynolds and Potter 1967).

The mean flow which is achieved in this way should then correspond to characteristic values $\tilde{\mathscr{R}}_m$ smaller than R_c, excluding the possibility of a supercritical equilibrium flow.

This relationship between the existence of subcritical instabilities and the absence of supercritical equilibrium flows is in agreement with the results of Reynolds and Potter (1967).

The argument does not eliminate the possibility of some more complex equilibrium flow where the disturbance would consist in the superposition of a large number of modes whose intricated action on the mean flow would actually modify it in the stabilizing sense, translating the characteristic surfaces $\mu_1^n(\alpha, \gamma)$ to higher μ_1 regions but then, such an equilibrium flow, as it would be progressively built up, with the corresponding loss of information — which we associate here with the Navier-Stokes equations since the Euler Lagrange equations of Serrin's problem are accepted approximations only for the stationary stations — would have all the characteristic of a turbulent flow and this mainly suggests that (statistically) steady turbulent flows could be incorporated in the present approach, somehow.

We shall examine this aspect in the next section.

4. STATISTICALLY STEADY TURBULENT CHANNEL FLOW

The postulates on which the Malkus theory is based have been much debated and different authors have often rephrased them to enlighten certain specific aspects (e. g. Spiegel 1962, Townsend 1962, Lumley 1965, Reynolds and Tiederman 1967, Nihoul 1967*b*). Malkus himself has never employed his fourth (variational) postulate in its original form. He actually inverted the variational principle and sought the minimum Reynolds number (associated with the marginal stability of the mean flow), holding Z constant.

Regarding the marginal stability of the mean flow, Reynolds and Tiederman (1967) have shown that indeed the *direct* influence of the « proper » turbulent field **w** on the birth and growth of a disturbance is negligible — slightly stabilizing in fact — and that the essential action of **w** is the modification of the mean velocity profile by the average turbulent Reynolds stresses. This result was also predicted by Nihoul (1967*a*) on the basis of a comparison between the mean velocity profile in turbulent

(*) *w* being of course differently normalized.

channel flow and the laminar Hartmann profile in magnetohydrodynamic channel flow under a large transverse magnetic field. Both types of profiles are known to be flat-topped with narrow regions of transition near the walls and — although the flattening agent is different, internally applied turbulent Reynolds stresses in the first case, externally applied Lorentz force in the second case — these profiles can be shown to be very stable. The increased stability is found to be due to the reduction of the mean flow scale in the boundary regions which extends the action of the inertial restoring forces up to such small scale disturbances that these are damped out by viscous dissipation. (Lin 1955, Nihoul 1967*a*). Although it is essential in the modification of the velocity profile, the direct action of the Lorentz force is found entirely negligible on the disturbance itself (Lock 1956).

According to the third hypothesis of Malkus, the rate of turbulent transport of momentum $\langle w_1 w_2 \rangle$ can be expanded in a finite series of functions of larger and larger « wave number » k_n ; the series being terminated at some smallest scale (i. e. some maximum wave number k_M).

Following Townsend (1962), we may visualize the different modes of motion contributing to the transport of momentum as motions obtaining energy directly from the mean flow and losing it by viscous dissipation and by non-linear transfer to a « background » of other turbulent motions (not transferring momentum) in such a way that the mode amplitudes are kept stationary in time. The non-linear transfer acts thus as a stabilizing influence ; this influence being negligible on the mode of highest order which is neutrally stable in consequence of the balance between energy transfer from the mean flow and energy loss by direct viscous dissipation.

The background is however not described by Malkus's theory in its simple form and in lieu of the actual modes imagined by Townsend one must seek then « equivalent » stationary modes whose combination comparable to a superposition of successive equilibrium disturbances on the mean flow provides the best approximation of the actual motion (*).

Rephrased as above, the Malkus theory seems the natural opening of the ideas put forward in the preceding sections. Indeed, to some extent, the statistically steady turbulent flow which is ultimately achieved — once secondary steady flows have failed to settle — is a form of equilibrium flow where the disturbance superimposed on the mean flow has become so intricatedly complex that — with the corresponding loss of information — its properties may only be known statistically (Landau 1944). One should expect, then, that, assimilating ensemble and space averages, many aspects of the description valid for equilibrium flows will also be pertinent to statistically steady turbulent flows.

In particular, it seems reasonable to inquire into the possibility of such flows being also extremizing solutions of a Reynolds number functional.

We note immediately that the minimum values of the Reynolds number (17) — where **w** denotes now the « proper » turbulent field —, subject to the constraints (2), (13), (18) and the boundary conditions (19), are obtained for eigensolutions of the Euler-Lagrange equations (3) and (12). For these solutions, the proprer turbulent energy E is stationary and the mean flow is given in terms of the average turbulent Reynolds stresses by the correct steady state equation (12).

The constraint (18) expresses Malkus's condition that Z — i. e., the ratio of

(*) Since the highest order mode in this description is not approximated, one should expect to obtain fairly goods results whenever the principal objective is the determination of the leading terms in the asymptotic solutions for large K_M (of the velocity profile away from the boundaries, for instance) (Malkus 1961).

the actual dissipation rate and the dissipation rate which would result from a parabolic flow with the same $\bar{u}$ — be held constant.

Also the expansion of the turbulent field in different scales components of the type mentioned above appears to be compatible, with the object of finding the smallest possible $\mathcal{R}$ for constant Z, only if the series is interrupted at some wave number k_M for each mean flow realization.

Indeed, let us compare the viscous dissipation in any turbulent component (say, of wave number K, and amplitude normalized to 1) to the energy released to it by a mean flow of a specific shape (which we expect flat-topped with narrow regions of transition near the walls). We observe that, once the component wavelength K^{-1} is sufficiently smaller than the width of the boundary regions of the mean velocity profile, the rate of energy transfer decreases rapidly —, it is expressed by an integral bearing on the product of two functions one of which oscillates many times over the region where the other, $\dot{u}$, is significantly different from zero — and becomes much smaller than the dissipation rate which we may expect on the contrary to increase with decreasing scale.

Since $\mathcal{R}$ is the ratio of the total dissipation rate and the total rate of energy transfer, it certainly will have a smaller value if wave number components which bring much larger contributions to the numerator than to the denominator are cut off of the spectrum. This requires the existence of a smallest scale k_M^{-1} and suggests that the corresponding highest mode is marginally stable on the mean flow since marginal stability is precisely the frontier beyond which dissipation dominates. The relation between the smallest scale of motion and the width of the transition region of the mean flow is in agreement with the results obtained by Malkus (1956) and Nihoul (1966).

A solution — appropriate to turbulence and to the present description — of the non-linear eigenvalue problem set by the Euler-Lagrange equations of the variational problem may be sought by expressing the proper turbulent field $\mathbf{w}$ in the following form proposed by Meksyn (1964)

(36) $$w_1 = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} [a_1^{l,m} \cos l\theta + b_1^{l,m} \sin l\theta] \cos m\gamma x_3$$

(37) $$w_2 = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} [a_2^{l,m} \cos l\theta + b_2^{l,m} \sin l\theta] \cos m\gamma x_3$$

(38) $$w_3 = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} [a_3^{l,m} \cos l\theta + b_3^{l,m} \sin l\theta] \sin m\gamma x_3$$

where θ is the same as in (20), (21) and (22) and where the a's and the b's are functions of x_2 related by the conditions of incompressibility

(39) $$\dot{a}_2^{l,m} = - l\alpha b_1 - m\gamma a_3$$

$$(l, m = 1, 2, 3, \ldots)$$

(40) $$\dot{b}_2^{l,m} = l\alpha a_1 - m\gamma b_3.$$

Introducing Lagrange multipliers as in section 3, an infinite set of Euler-Lagrange equations is obtained and it is readily seen that for each l and m they have the same form as eqs (26) to (31) where α is replaced by $\overline{\alpha} = l\alpha$ and γ by $\overline{\gamma} = m\gamma$. All these equations combine to yield equation (3) with

$$\lambda = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (\lambda_1^{l,m} \cos l\theta + \lambda_2^{l,m} \sin l\theta) \cos m\gamma x_3. \tag{41}$$

In addition there is a Euler-Lagrange equation for u which can be brought in the form (12) exactly as we did in section 3. This equation, together with the whole system of equations for the a's and b's and the boundary conditions, sets a non-linear eigenvalue problem for μ_1.

Considering first the linear eigenvalue problem associated with eq. (26) to (31) where α is replaced by $\overline{\alpha} = l\alpha$ and γ by $\overline{\gamma} = m\gamma$ and where u is the actual mean turbulent velocity profile, we may expect that the eigenvalues $\mu_1^n(\overline{\alpha}, \overline{\gamma})$ are all points of the surfaces tentatively described in section 3 and sketched in figure 1 ; these surfaces being transported to higher μ_1 regions in consequence of the greater stability of the flat-topped mean velocity profile. Let the smallest eigenvalue of the non-linear problem be μ_1^*. This should not be the smallest μ_1^n on the surfaces above because then the infinite set of Euler-Lagrange equations for the $a^{l,m}$ and $b^{l,m}$ would only admit one non trivial solution of a specific wave number vector $(\overline{\alpha}, n, \overline{\gamma})$, definitely unable to produce the required reshaping of the mean velocity profile. We expect instead μ_1^* to be the smallest eigenvalue associated with some high transverse mode n^* whose scale is presumably related to the width of the mean flow boundary layers (to be determined) (see fig. 1).

This situation would be appropriate to the present description since then the n^*-mode is marginally stable on the mean flow, all higher modes are absent from the series expansions (36), (37) and (38) as the surfaces $\mu_1^n(\overline{\alpha}, \overline{\gamma})$ are entirely on the right-hand side of the plane $\mu_1^n = \mu_1^*$ for $n > n^*$, all the modes of lower order $(n < n^*)$ are stationary, their respective wave numbers $\overline{\alpha}$ and $\overline{\gamma}$ being distributed at the intersections of their characteristic surfaces $\mu_1^n(\overline{\alpha}, \overline{\gamma})$ and $\mu_1^n = \mu_1^*$.

The question which remains to be answered concerns the marginal stability to infinitesimal disturbances of the mean velocity profile regarded as a laminar motion (i. e. solely holding the dialogue with the perturbation). The existence in the proper turbulent spectrum of wave number components for which the smallest eigenvalue of marginal stability is smaller than the actual Reynolds number of the flow would seem to suggest that infinitesimal disturbances might occur at those scales and be amplified. However such « large scale » perturbations (*), being of infinitesimal amplitude, could not resist the restoring force exerted by the mean flow (Nihoul 1967*a*) (**) and would be — the viscous dissipation defaulting — the victims of inertial stabilization.

This remark completes the present discussion which it is hoped may reveal additional arguments in favor of the Malkus theory and the variational approach to turbulent channel flows.

(*) Larger than the width of the mean flow boundary regions.
(**) The corresponding turbulent components are maintained against the inertial restoring of the mean flow by the contra-inertial forces which — being of comparable amplitude — they may develop.

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to the Comité National de Mécanique de l'Académie Royale de Belgique for stimulating this research and to Professor W. V. R. Malkus for his sustained interest in its progress.

REFERENCES

GILBERT J. E., KNOPS R. J., (1967), *Arch. Rat. Mech. Anal.*, **25**, 271.

GORTLER H., KIRCHGÄSSNER K., SORGER P., VELTE W., (1965), *Nova Acta Leopoldina*, N. F., **30**, 23.

KIRCHGÄSSNER K., (1960), *Arch. Rat. Mech. Anal.*, **6**, 20.

KIRCHGÄSSNER K., (1961), *ZAMP*, **XII**, 14.

KNOPS R. J., WILKES E. W., (1966), *Int. J. Engng. Sci.*, **4**, 303.

LANDAU L., (1944), *C. R. Acad. Sc. URSS*, **44**, 311.

LIAPOUNOV M. A., (1947), Problème Général de la Stabilité du Mouvement, Princeton, U. P.

LIN C. C., (1955), The Theory of Hydrodynamic Stability, Cambridge U. P.

LOCK R. C., (1956), *Proc. Roy. Soc. London* (A), **233**, 105.

LUMLEY J. L., (1965), Atmospheric Turbulence and Radio Wave Propagation. *Proceedings of the International Colloquium*, Moscow, June 15-22.

MALKUS W. V. R., (1956), *J. Fluid. Mech.*, 1, 521.

MALKUS W. V. R., (1961), *Suppl. Nuovo Cimento*, **22**, X, 376.

MEKSYN D., (1964), *Zeit. für Phys.*, **178**, 159.

MOVCHAN A. A., (1959), *Appl. Math. Mech. NY*, **23**, 483.

MOVCHAN A. A., (1960), *Appl. Math. Mech. NY*, **24**, 988.

MOVCHAN A. A., (1963), *Archiom Mech. strosow*, **15**, 659.

NEHARI Z., (1960), *Trans. Amer. Math. Soc.*, **95**, 101.

NEHARI Z., (1961), *Acta Math.*, **105**, 141.

NIHOUL J. C. J. , (1966), *J. Fluid. Mech.*, **25**, 1.

NIHOUL J. C. J., (1967*a*), *Rev. Math. Pures et Appl.*, **10**, XII, 1503.

NIHOUL J. C. J., (1967*b*), *J. de Mécanique*, **6**, 171.

NIHOUL J. C. J., (1968), *Z.A.M.M.* to be published.

ORR W. McF, (1907), *Proc. Roy. Irish Acad.* (A), 27, 69.

PRITCHARD A. J., (1968), *J. Inst. Maths Applics*, **4**, 78.

REYNOLDS O., (1895), *Phil. Trans. Roy. Soc. London* (A), **186**, 123.

REYNOLDS W. C., POTTER M. C., (1967), *J. Fluid. Mech.*, **27**, 465.

REYNOLDS W. C., TIEDERMAN W. G., (1967), *J. Fluid. Mech.*, **27**, 253.

SERRIN J., (1959), *Arch. Rat. Mech. Anal.*, **3**, 1.

SPIEGEL E. A., (1962), *Mécanique de la turbulence*, Ed. du CNRS, Paris, p. 181.

STUART J. T., (1958), *J. Fluid. Mech.*, **4**, 1.

STUART J. T. (1962), *Proc. of the 10th international congress of applied math.* Stresa, (Italy), 1960, Elsevier.

TOWNSEND A. A., (1962), *Mécanique de la turbulence*, Ed. du C.N.R.S., Paris, p. 166.

VAINBERG M. M., TRENOGIN V. A., (1962), *Russ. Math. Surv.*, **17**, 1.

ZUBOV V. I., (1964), Methods of A. M. Liapounov and their application Noordhoff.