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ON THE VARIETIES $P_{2\mathrm{m},0}$ OF OCKHAM ALGEBRAS

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ABSTRACT. We consider the varieties $\boldsymbol{P}_{2m,0}$ of Ockham algebras, i.e., those in which for any element x of the dual space X of the algebra, the equality $g^{2m}(x) = x$ holds. We use the same method as in [5] and show that these varieties have specific properties and consequently are more tractable than the others. In particular, all the subvarieties of $\boldsymbol{P}_{2m,0}$ can be characterised by axioms that are self-dual. We apply these properties to the description of $\boldsymbol{P}_{6,0}$ and $\boldsymbol{P}_{10,0}$.

1. Introduction

This paper is a sequel to [5] and we assume familiarity with it. Most notions that we are dealing with can be found in [5], where the reader is given a complete bibliography.

We just recall that an *Ockham algebra* is an algebra $(A; \vee, \wedge, \sim, 0, 1)$ of type $(2,2,1,0,0)$ such that $(A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\sim$ is a dual endomorphism:

$$\sim(a \vee b) = \sim a \wedge \sim b, \quad \sim(a \wedge b) = \sim a \vee \sim b, \quad \sim 0 = 1, \quad \sim 1 = 0\ ,$$

and that the space X dual to A is endowed with an order-reversing and continuous map g.

We denote the variety of Ockham algebras by $\boldsymbol{O}$. Among the subvarieties of $\boldsymbol{O}$ are the subvarieties $\boldsymbol{P}_{m,n}$ $(m > n \geq 0)$ defined by

$$A \in \boldsymbol{P}_{m,n} \iff g^m(x) = g^n(x), \quad \forall x \in X\ .$$

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Particularly interesting are the subvarieties $\boldsymbol{P}_{2m,0}$ which are considered here. In these subvarieties the mapping $g\colon X \to X$ is an involution of period $2m$; if $m = 1$, we obtain the variety $\boldsymbol{P}_{2,0}$ of de Morgan algebras.

We develop a method for describing $\Lambda(\boldsymbol{P}_{2m,0})$, the lattice of subvarieties of $\boldsymbol{P}_{2m,0}$, and for giving each subvariety of $\boldsymbol{P}_{2m,0}$ an adequate equational basis. The method does not differ fundamentally from the one that is used in [5] but offers many simplifications that clarify the problem. Indeed, the subvarieties $\boldsymbol{P}_{2m,0}$ have remarkable properties that are not shared by the other $\boldsymbol{P}_{m,n}$. It is mainly due to the fact that the map g is a bijection.

The subdirectly irreducible algebras of $\boldsymbol{P}_{2m,0}$ are simple (in other words, every variety $\boldsymbol{P}_{2m,0}$ is *semisimple*) and have been described in [2] and [4]. We give a brief outline.

Let $(X, \leq, g)$ be the dual space of a subdirectly irreducible algebra which properly belongs to $\boldsymbol{P}_{2m,0}$, that is, which belongs to $\boldsymbol{P}_{2m,0}$ but to no strictly smaller class, and denote the elements of X by $0, 1, ..., 2m-1$. The order-reversing map g is defined by

$$g(i) = \begin{cases} i+1 & \text{if } i \in \{0, 1, ..., 2m-2\} ; \\ 0 & \text{if } i = 2m-1 . \end{cases}$$

The subset $\Gamma(0) = \{x \in X \mid 0 \prec x,\ x \leq m\}$ determines the whole structure of X. If $\Gamma(0) = \{r_1, ..., r_k\}$ then the number of connected components of X is

$$t = \gcd(m, r_1, ..., r_k) .$$

It follows that X is connected if and only if $m, r_1, ..., r_k$ are coprime.

Every connected component of X is either a generalised crown or a singleton.

The cardinality of $Si(\boldsymbol{P}_{2m,0})$ has been determined in Theorem 2.25 of [6]:

$$|Si(\boldsymbol{P}_{2m,0})| = D(2m) + \sum_{p|m} \Psi(p)$$

where $D(2m)$ is the number of divisors of $2m$ and $\Psi(p) = 2^{[\frac{p+1}{2}]} - 1$.

The number α_m of non-isomorphic simple Ockham algebras that *properly* belong to $\boldsymbol{P}_{2m,0}$ is given by

$$\alpha_m = \begin{cases} 2^{m/2} & \text{if } m \text{ is even} ; \\ 2^{(m+1)/2} & \text{if } m \text{ is odd} . \end{cases}$$

The paper is organised as follows. In section 2 we show that all the g-relations on the dual space X, i.e., all the relations $g^i(x) \geq g^j(x)$ for a given $x \in X$, can be written in a canonical form:

$$\begin{array}{lll} g^i \geq g^0 \text{ or } g^0 \geq g^i & \text{for } i \text{ odd}, & 1 \leq i \leq m ; \\ g^i = g^0 & \text{for } i \text{ even}, & 2 \leq i \leq m \text{ and } i \mid 2m . \end{array}$$

We determine the partially ordered set of the implications that link the g-relations, so obtaining for each $\boldsymbol{P}_{2m,0}$ the number of non-equivalent g-relations.

We also analyse the consequences of the conjunction of two g-relations. We are mainly interested in the following opposite cases:

– the conjunction of two g-relations implies any g-relation definable on X;
– the conjunction of two g-relations does not yield any new g-relation.

The above considerations enable to construct the "Cayley table" of the g-relations. Then we turn our attention to the g-relations that are satisfied by *all* the elements, as well as to the disjunction of a finite number of such g-relations. These disjunctions of g-relations are called axioms and have the strong property of being self-dual.

Sections 3 and 4 contain two illustrative examples: $\boldsymbol{P}_{6,0}$ and $\boldsymbol{P}_{10,0}$. The choice of the first example is justified by the fact that this variety has been already studied by M. Ramalho in [3], so enabling to compare results obtained by different ways. The analysis of $\boldsymbol{P}_{10,0}$ reveals how things are getting more complex when m is increasing, as is shown by the following table:

characteristics of the variety	$\boldsymbol{P}_{6,0}$	$\boldsymbol{P}_{10,0}$
number of non-equivalent g-relations	5	7
number of non-equivalent axioms	14	42
number of subdirectly irreducibles	8	12
number of subvarieties (including the trivial one)	20	70

2. Main properties

In $\boldsymbol{P}_{2m,0}$ the mapping $g\colon X \to X$ is a bijection with $g^{-1} = g^{2m-1}$. We may define g^s with $s < 0$:

$$g^s = (g^{-1})^{-s} = (g^{-s})^{-1} .$$

Clearly, g^{-1} is order-reversing; more generally, g^s is order-reversing if and only if $|s|$ is odd and g^s is order-preserving if and only if $|s|$ is even.

Very often we omit the letter x and write, for instance, $g^5 \geq g^0$ instead of $g^5(x) \geq g^0(x)$.

Theorem 1. *Let i be even and $i \in \{2, 4, ..., 2m-2\}$. In $\boldsymbol{P}_{2m,0}$ we have*

$$g^i \geq g^0 \iff g^0 \geq g^i \iff g^i = g^0 \iff g^k = g^0 ,$$

where $k = \gcd(2m, i)$.

Proof: Consider any g-relation $g^i \geq g^0$ with even i and $i \leq 2m$. Then

$$g^0 \leq g^i \leq g^{2i} \leq ... \leq g^{2mi} = g^0 ,$$

hence $g^0 = g^i$.

Similarly, from $g^0 \geq g^i$ follows $g^0 = g^i$. So we have

$$g^i \geq g^0 \iff g^0 \geq g^i \iff g^i = g^0 .$$

Now let $k = \gcd(2m, i)$. Suppose $g^k = g^0$. So there is $s \in \mathbb{N}$ such that $i = s\,k$, hence $g^i(x) = g^{sk}(x) = x$.

Conversely, suppose that $g^i(x) = x$. By the Bezout identity, there are $s, t \in \mathbb{Z}$ such that $k = 2m\,s + i\,t$. So

$$g^k(x) = g^{2ms}\, g^{it}(x) = g^{2ms}(x) = x \text{ . } \Diamond$$

Theorem 2. *In $\boldsymbol{P}_{2m,0}$ every g-relation is equivalent to a g-relation of the form*

$$\begin{array}{llll} & g^i \geq g^0 & \textit{with } i \textit{ odd}, & 1 \leq i \leq m\ , \\ \textit{or} & g^0 \geq g^i & \textit{with } i \textit{ odd}, & 1 \leq i \leq m\ , \\ \textit{or} & g^i = g^0 & \textit{with } i \textit{ even}, & 2 \leq i \leq m, \quad i \,|\, 2m\ . \end{array}$$

Proof: Consider any g-relation $g^j \geq g^k$ with $0 \leq j, k \leq 2m-1$ and $j \neq k$.

If j is even, then $g^{-j}\, g^j \geq g^{-j}\, g^k$, that is $g^0 \geq g^{k-j} = g^s$ with $s \equiv k - j \pmod{2m}$ and $1 \leq s < 2m$. If $s \leq m$, then $g^j \geq g^k$ is equivalent to $g^0 \geq g^s$ with $1 \leq s \leq m$. If $s > m$, then $1 \leq 2m - s < m$ and we have

$$g^0 \geq g^s \iff g^{2m-s} \geq g^0 \quad \text{if } s \text{ is even}$$

and

$$g^0 \geq g^s \iff g^{2m-s} \leq g^0 \quad \text{if } s \text{ is odd} \quad .$$

If j is odd, then $g^{-j}\, g^j \leq g^{-j}\, g^k$, that is $g^0 \leq g^{k-j}$ and the proof goes on similarly to the previous case.

Now suppose that x satisfies $g^k(x) \geq g^0(x)$ or $g^0(x) \geq g^k(x)$ for k even, $1 \leq k \leq m$. By Theorem 1, x satisfies $g^i(x) = g^0(x)$ where $i = \gcd(2m, k)$, and consequently i is even, $2 \leq i \leq m$ and $i \,|\, 2m$. $\Diamond$

In what follows we shall use the notations $[i]$ and $[i']$, $1 \leq i \leq m$, for the g-relations $g^i \geq g^0$ and $g^0 \geq g^i$, respectively. Of course, in case i is even, $[i]$ will mean $g^i = g^0$.

In $\boldsymbol{P}_{2m,0}$ the g-relations are not completely independent and some implications between them exist, at least when the exponents of g are even.

Theorem 3. *Let $L \in \boldsymbol{P}_{2m,0}$ and $1 \leq i, j \leq m$.*

(1) *If i is odd, then each $[i]$ and $[i']$ is independent of all the other g-relations.*
(2) *If i is even and divides $2m$, then $[i]$ is independent of all the g-relations $[j]$ and $[j']$ with j odd.*
(3) *If i and j are even and divide $2m$, then $[i] \Rightarrow [j]$ if and only if i divides j.*

Proof: To prove (1) it suffices to exhibit an algebra $L \in \boldsymbol{P}_{2m,0}$ whose dual space X has an element that satisfies $[i]$ and no other g-relation and proceed similarly for $[i']$. Let L be the simple Ockham algebra whose dual space $X = \{0, 1, ..., 2m-1\}$ satisfies $\Gamma(0) = \{i\}$ with i odd, $1 \leq i \leq m$. The space X has $t = \gcd(m, i)$ connected components each of which is a $\frac{m}{t}$-crown. Whatever value t may have, the element 0 is covered exactly by i and $2m - i$. Hence for any j such that $1 \leq j \leq m$, $j \neq i$, the g-relation $[j]$ is not satisfied by the element 0. Similarly, any $[k']$ with k odd, $1 \leq k \leq m$, and any $[l]$ with l even are not satisfied by the element 0.

To prove that $[i']$ is independent of all the other g-relations, we consider the same algebra L. The element $2m - i$ of its dual space satisfies $[i']$ and none of the other g-relations.

The proof of (2) follows exactly the same way. Let i be even, $2 \leq i \leq m$ and $i \,|\, 2m$. Consider the antichain $X = \{0, 1, ..., i-1\}$ with

$$g(x) = \begin{cases} x+1 & \text{if } 0 \leq x < i-1 \,; \\ 0 & \text{if } x = i-1 \,. \end{cases}$$

The dual algebra L belongs to $\boldsymbol{P}_{i,0} \subseteq \boldsymbol{P}_{2m,0}$, satisfies $[i]$ but no g-relation $[j]$ or $[j']$ for j odd.

Finally, let i, j be even and such that $2 \leq i, j \leq m$, $i \,|\, j$ and $j \,|\, 2m$. It follows that $j = i\,k$ for some k. If x satisfies $[i]$, it also satisfies $[j]$ since $g^i(x) = x$ implies $g^{ik}(x) = x$, i.e., $g^j(x) = x$. Conversely, if $[i] \Rightarrow [j]$, then $i \,|\, j$. In fact, consider again the antichain $X = \{0, 1, ..., i-1\}$ with g as in (2). The element 0 satisfies $[i]$ and no other relation, except those $[j]$ for which $j = k\,i \leq m$. ◊

Corollary. *In $\boldsymbol{P}_{2m,0}$ the number of non-equivalent g-relations is*

$$\begin{array}{ll} m + D(m) & \textit{if } m \textit{ is odd}\,, \\ m + D(m) - 1 & \textit{if } m \textit{ is even}\,, \end{array}$$

where $D(m)$ is the number of divisors of m.

Proof: Suppose that m is odd. We have

$$\left|\left\{i\colon\ i \text{ odd. } 1 \leq i \leq m\right\}\right| = \frac{m+1}{2} \;;$$

$$\left|\left\{i\colon\ i \text{ even, } 2 \leq i \leq m,\ i \,|\, 2m\right\}\right| = \left|\left\{i\colon\ i = 2s,\ s \neq m,\ s \,|\, m\right\}\right| = D(m) - 1 \,.$$

The number of non-equivalent g-relations is $m + D(m)$.

Now let m be even. We have

$$\left|\left\{i\colon\ i \text{ odd, } 1 \leq i < m\right\}\right| = \frac{m}{2} \;;$$

$$\left|\left\{i\colon\ i \text{ even, } 2 \leq i \leq m,\ i \,|\, 2m\right\}\right| = D(m) - 1 \,.$$

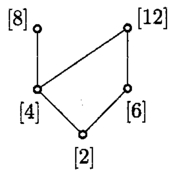
The number of non-equivalent g-relations is $m + D(m) - 1$. ◊

Examples. In $\boldsymbol{P}_{22,0}$ there are 13 non-equivalent g-relations:

[1] [1'] [3] [3'] ... [11] [11'] [2]

In $\boldsymbol{P}_{24,0}$ there are 17 non-equivalent g-relations:

[1] [1'] [3] [3'] ... [11] [11']



The interplay of the g-relations on the dual space X can be clarified by considering the conjunction of all pairs of such g-relations: if an element $x \in X$ satisfies two g-relations, say R_1 and R_2, does it satisfy other g-relations? Not surprisingly three cases are possible:

1) x satisfies R_1 and R_2 but no other g-relation;
2) x satisfies all g-relations;
3) x satisfies some g-relations in addition to R_1 and R_2.

The first two cases are particularly interesting and it is worth giving them a special name.

A pair of g-relations will be said *sterile* if the conjunction of these g-relations does not imply any new g-relation. On the contrary, a pair of g-relations will be said *fruitful* if the conjunction of these g-relations implies all the g-relations definable in $\boldsymbol{P}_{2m,0}$. For brevity we shall write $[i, j]$ instead of $[i]$ & $[j]$ and improperly $[i, j]$ will be called a pair.

The following lemma is obvious.

Lemma. *Let $A \in \boldsymbol{P}_{2m,0}$ and $m \geq i > j \geq 1$.*
If j is even, then $g^i \geq g^j \Leftrightarrow [i-j]$ and $g^j \geq g^i \Leftrightarrow [(i-j)']$.
If j is odd, then $g^i \geq g^j \Leftrightarrow [(i-j)']$ and $g^j \geq g^i \Leftrightarrow [i-j]$.
The following table indicates some immediate consequences of the conjunction of two g-relations among $\{[i], [i'], [j], [j']\}$ which are indicated at the top of the relevant column.

$m \geq i > j \geq 1$	$[i, j]$	$[i', j']$	$[i, j']$	$[i', j]$
i, j odd			$[i-j]$	$[i-j]$
i odd, j even	$[i-j]$	$[(i-j)']$	$[i-j]$	$[(i-j)']$
i even, j odd	$[i-j]$	$[(i-j)']$	$[(i-j)']$	$[i-j]$
i, j even	$[i-j]$	$[i-j]$	$[i-j]$	$[i-j]$

Theorem 4. *Let $A \in \boldsymbol{P}_{2m,0}$.*
(1) *$[m, m']$ is sterile.*
(2) *If i, j are odd and $m \geq i > j \geq 1$, then $[i, j]$ and $[i', j']$ are sterile.*

Proof: In every case it suffices to exhibit an example in which the pair in question is sterile.

(1) Take the simple algebra whose dual space is the antichain $\{0, 1, ..., m-1\}$. This algebra belongs to $\boldsymbol{P}_{m,0} \subset \boldsymbol{P}_{2m,0}$. Clearly $g^m(0) = 0$ and the element 0 satisfies $[m]$ and $[m']$ solely.

(2) Take the simple algebra that properly belongs to $\boldsymbol{P}_{2m,0}$ and in which $\Gamma(0) = \{i, j\}$. Clearly, the element 0 satisfies $[i]$ and $[j]$, but no other g-relation, and the element $2m-i$ satisfies $[i']$ and $[j']$ but no other g-relation. ◊

Now we determine some fruitful pairs of g-relations.

Theorem 5. *Let $A \in \boldsymbol{P}_{2m,0}$. If $i < m$ is odd and coprime with m, then $[i, i']$ is fruitful.*

Proof: We have

$$g^i \geq g^0 \;\;\&\;\; g^0 \geq g^i \quad \Longrightarrow \quad g^0 = g^i = g^{ni}, \quad \forall n \in \mathbb{Z} .$$

Since i is coprime with $2m$, there are $p, q \in \mathbb{Z}$ such that $p\,i = 2m\,q + 1$. It follows that $g^{pi} = g^{2mq+1} = g$.

Since $g^{pi} = g^0$, we have $g^0 = g$, all powers of g are equal and $[i, i']$ is fruitful. ◊

Theorem 6. *Let $A \in \boldsymbol{P}_{2m,0}$. Let also i, j be odd and such that $m \geq i > j > 0$. If j is coprime with m or if i is coprime with j, then $[i, j']$ and $[i', j]$ are fruitful.*

Proof: Let j be coprime with m. By the Lemma, $[i, j'] \Rightarrow [i - j]$ and $[i, i - j] \Rightarrow [i - i + j] = [j]$. By Theorem 5, $[j, j']$ is fruitful. It follows that $[i, j']$ is also fruitful. In a quite similar way one can show that $[i', j]$ is fruitful.

Now let j be coprime with i. Since $[i, j'] \Rightarrow [i - j]$, we have

$$g^0 = g^{i-j} = g^{n(i-j)}, \qquad \forall n \in \mathbb{Z} .$$

Moreover, $[i, i - j] \Rightarrow [j]$ and $[j, j'] \Rightarrow g^0 = g^j = g^{nj}$, $\forall n \in \mathbb{Z}$. From $\gcd(i, j) = 1$ follows $\gcd(i - j, j) = 1$. So there are $p, q \in \mathbb{Z}$ such that $pj = q(i - j) + 1$. We thus have

$$g^0 = g^{pj} = g^{q(i-j)+1} = g^0\, g^1 = g .$$

All powers of g are equal and $[i, j']$ is fruitful.

The proof of $[i', j]$ fruitful is quite similar. ◊

Corollary. *Let $A \in \boldsymbol{P}_{2m,0}$ with m prime and let i, j be odd with $m \geq i \geq j > 0$. Then all $[i, j']$ and $[i', j]$ are fruitful, except if $i = j = m$.*

Proof: If $i = j = m$, then $[i, j'] = [i', j] = [m, m']$ which is sterile, by Theorem 4.

If $i = j < m$, then $[i, j'] = [i', j] = [i, i']$ with i odd and coprime with m and we apply Theorem 5.

If $m \geq i > j > 0$, then j is coprime with m and it suffices to apply Theorem 6. ◊

If one element of a pair of g-relations is odd and the other is even, then the following theorem can be applied.

Theorem 7. *Let $A \in \boldsymbol{P}_{2m,0}$, $2k + 1 \leq m$, $2n \leq m$ and $n|m$. Then, for every $t \in \mathbb{N}$ such that $|2(k - tn) + 1| \leq m$, $[2k + 1, 2n]$ implies $[|2(k - tn) + 1|]$ and $[(2k + 1)', 2n]$ implies $[|2(k - tn) + 1|']$.*

Proof: Note first that the necessity of the assumptions on k and n follow from Theorem 2.

We have

$$\begin{aligned}[2k+1,2n] &\iff g^{2k+1} \geq g^0 \ \&\ g^0 = g^{2n} \\ &\implies g^{2k+1} \geq g^0 \ \&\ g^0 = g^{2tn} \quad (t \in \mathbb{N}) \\ &\implies g^{2k+1} \geq g^{2tn} \\ &\iff g^{2(k-tn)+1} \geq g^0 \\ &\iff g^{|2(k-tn)+1|} \geq g^0 \quad \text{since } 2(k-tn)+1 \text{ is odd}.\end{aligned}$$

The algebra A satisfies $[|2(k-tn)+1|]$ for every t such that $|2(k-tn)+1| \leq m$. That $[(2k+1)', 2n]$ implies $[|2(k-tn)+1|']$ is proved in a similar way. ◊

Example. Let $A \in \boldsymbol{P}_{12,0}$ and $k=2$. Since $2n \leq m$ and $n|m$, the possible values of n are $1, 2, 3$.

If $n=1$, the condition $|2(2-t)+1| \leq 6$ is satisfied by $t=1$, $t=2$, and yields $[5,2] \Rightarrow [3]$, $[5,2] \Rightarrow [1]$, respectively.

If $n=2$, the condition $|2(2-2t)+1| \leq 6$ is satisfied by $t=1$, $t=2$, and yields $[5,4] \Rightarrow [1]$, $[5,4] \Rightarrow [3]$, respectively.

If $n=3$, the condition $|2(2-3t)+1| \leq 6$ is satisfied by $t=1$ and yields $[5,6] \Rightarrow [1]$.

Theorems 4–7 are very useful for the construction of the Cayley table of the g-relations, as it will be seen in the examples $\boldsymbol{P}_{6,0}$ and $\boldsymbol{P}_{10,0}$ which follow.

Now we turn our attention to the g-relations that are satisfied by *all* the elements of the dual space and for which we use parentheses instead of brackets. For instance, the notation $(3')$ will mean that *all* the elements x of the dual space X satisfy $x = g^0(x) \geq g^3(x)$. Such g-relations, as well as the disjunction of a finite number of them, will enable us to define all the subvarieties of $\boldsymbol{P}_{2m,0}$ and therefore receive the name of *axioms.* As an example, $(11'52)$ means that all the elements x of X satisfy $g(x) \geq x$ or $x \geq g(x)$ or $g^5(x) \geq x$ or $g^2(x) = x$.

The subvarieties $\boldsymbol{P}_{2m,0}$ enjoy the following interesting property.

Theorem 8. *In $\boldsymbol{P}_{2m,0}$ all the axioms (as defined above) are self-dual.*

Proof: Every axiom is the disjunction of g-relations of the form $x \geq g^i(x)$ and $g^j(x) \geq x$. Hence it can be written

(1) $\quad (\forall x \in X) \quad x \geq g^{i_1}(x) \vee x \geq g^{i_2}(x) \vee \ldots \vee g^{j_1}(x) \geq x \vee g^{j_2}(x) \geq x \vee \ldots$

By substituting $g(x)$ to x in (1) we obtain

$$(\forall x \in X) \quad g(x) \geq g^{i_1+1}(x) \vee g(x) \geq g^{i_2+1}(x) \vee \ldots \vee g^{j_1+1}(x) \geq g(x) \vee g^{j_2+1}(x) \geq g(x) \vee \ldots$$

from which we deduce that

$$(\forall x \in X) \quad g^{2m-1} g(x) \leq g^{2m-1} g^{i_1+1}(x) \vee \ldots \vee g^{2m-1} g^{j_1+1}(x) \leq g^{2m-1} g(x) \vee \ldots,$$

that is

(2) $\quad (\forall x \in X) \quad x \leq g^{i_1}(x) \vee x \leq g^{i_2}(x) \vee \ldots \vee g^{j_1}(x) \leq x \vee g^{j_2}(x) \leq x \vee \ldots$

Clearly (2) is the dual of (1). ◊

Theorem 8 reduces the number of non-equivalent axioms by half since for any $i, j, \ldots$ odd we have

$$(i') = (i)\,, \quad (i\,j) = (i'j')\,, \quad (i'j) = (i\,j')\,,$$

and if k is even

$$(i\,j\,k) = (i'j'k)\,, \quad (i'j\,k) = (i\,j'k)\,.$$

We lay stress on the fact that in the axioms the juxtaposition of the letters means *disjunction*, whereas in the g-relations a comma means *conjunction*.

Theorem 9. *In $\boldsymbol{P}_{2m,0}$ the number of variables in every $\sim$-inequality can be reduced to 1 or 2. It can be reduced to 1 if the notation of the axiom involves no dashed odd number or involves, apart from even numbers, only dashed odd numbers.*

Proof: By Theorem 1, if i is even, $g^i \geq g^0$ is equivalent to $g^0 \geq g^i$. Thus in the tabulation, the row corresponding to $[i]$ can be written with 0 in the first column and i in the fourth, or vice-versa.

If the odd numbers involved in the axiom are all dashed, all rows in the tabulation can be written with 0 in the first column. Applying Theorem 5.3 of [1], we get a $\sim$-inequality with one variable. Similarly, writing the rows corresponding to the even numbers with 0 in the fourth column, if the axiom involves no dashed odd number, we get a $\sim$-inequality with one variable. It is now obvious that with a similar reasoning we conclude that in $\boldsymbol{P}_{2m,0}$ the number of variables in every $\sim$-inequality can be reduced to 1 or 2. ◊

Example. Consider the axiom $(11'33'2)$ in $\boldsymbol{P}_{6,0}$. Its tabulation is

$$\left|\begin{matrix} - & - & 1 & 0 \\ 0 & 1 & - & - \\ - & - & 3 & 0 \\ 0 & 3 & - & - \\ 0 & - & - & 2 \end{matrix}\right|\,.$$

In the 1st column, on the 2nd, 4th and 5th rows, we have the same digit 0. In the 4th column, on the 1st and 3rd rows, we also have the same digit 0. Hence two variables, say a and b, are sufficient to define the corresponding $\sim$-inequality:

$$a \wedge \sim a \wedge \sim^3 a \;\leq\; \sim^2 a \vee b \vee \sim b \vee \sim^3 b\,.$$

3. The subvariety $\boldsymbol{P}_{6,0}$

Step 1. By the Corollary of Theorem 3, there are in $\boldsymbol{P}_{6,0}$ five non-equivalent g-relations:

$$[1]\; g \geq g^0; \quad [1']\; g^0 \geq g; \quad [3]\; g^3 \geq g^0; \quad [3']\; g^0 \geq g^3; \quad [2]\; g^2 = g^0\,.$$

Ordered by implication, they form an antichain:

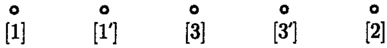


FIGURE 1

Step 2. Using Theorems 4–7 we obtain the following table:

&	[1]	[1′]	[3]	[3′]	[2]
[1]	⋱	A	/	A	[3]
[1′]		⋱	A	/	[3′]
[3]			⋱	/	[1]
[3′]				⋱	[1′]
[2]					⋱

TABLE 1

We recall that the letter A means the set of all non-equivalent g-relations, so it is used for the fruitful pairs. The sign / means "nothing new" and is used for the sterile pairs.

Step 3. Using Theorem 8 and some elementary considerations we show that there are 14 non-equivalent axioms in $\boldsymbol{P}_{6,0}$ (the equivalences due to the fact that the axioms are self-dual are indicated by the sign $\stackrel{d}{=}$):

$(1) \stackrel{d}{=} (1')$; $\quad$ $(2) = (12) \stackrel{d}{=} (1'2)$;

$(3) \stackrel{d}{=} (3') = (13) \stackrel{d}{=} (1'3')$; $\quad$ $(32) \stackrel{d}{=} (3'2) = (132) \stackrel{d}{=} (1'3'2)$;

$(11')$; $\quad$ $(11'2)$;

$(13') \stackrel{d}{=} (1'3)$; $\quad$ $(13'2) \stackrel{d}{=} (1'32)$;

$(33') = (133') \stackrel{d}{=} (1'33')$; $\quad$ $(33'2) = (133'2) \stackrel{d}{=} (1'33'2)$;

$(11'3) \stackrel{d}{=} (11'3')$; $\quad$ $(11'32) \stackrel{d}{=} (11'3'2)$;

$(11'33')$; $\quad$ $(11'33'2)$.

The above equivalences are easy to establish. For instance, since $(12) \stackrel{d}{=} (1'2)$, we have $(12) = (12)\ \&\ (1'2)$ and, owing to $[1, 1'] \Rightarrow [2]$, we obtain $(2) = (12)$. Similarly, from $(132) \stackrel{d}{=} (1'3'2)$ we deduce $(132) = (132)\ \&\ (1'3'2)$, hence $(132) \Rightarrow (32)$ since $[1, 1']$ and $[1, 3']$ are both fruitful.

The implications between these axioms are shown in the poset of Figure 2.

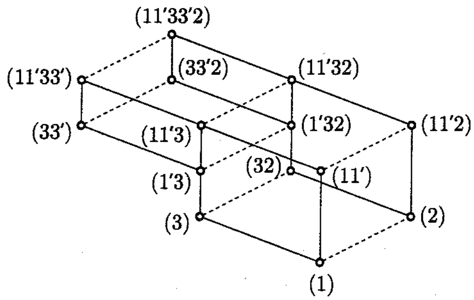


FIGURE 2

The axiom (1) implies all the other axioms since (1) means $x = g(x)$, $\forall x \in X$, hence $x = g(x) = g^2(x) = \dots$.

Since $(33') = (1'33')$, we have $(1'3) \Rightarrow (33')$.

The other implications are obvious.

In the ordered set of Figure 2 six elements are meet-irreducible; they form the subset

$$M = \Big\{(11'33'2),\ (11'33'),\ (33'2),\ (11'32),\ (11'2),\ (32)\Big\}\ .$$

All the other elements except (1) and (3), can be expressed as conjunctions of elements of M:

$$\begin{aligned}
(11'3) &= (11'33')\ \&\ (11'32)\ ;\\
(1'32) &= (11'32)\ \&\ (33'2)\ ;\\
(33') &= (11'33')\ \&\ (33'2)\ ;\\
(11') &= (11'33')\ \&\ (11'2)\ ;\\
(1'3) &= (11'33')\ \&\ (1'32)\\
&= (11'33')\ \&\ (11'32)\ \&\ (33'2)\ ;\\
(2) &= (11'2)\ \&\ (32)\ .
\end{aligned}$$

Moreover we have

$$(1) = (2)\ \&\ (3) = (11'2)\ \&\ (32)\ \&\ (3)\ .$$

Note also (see Table 2) that the simple algebra dual to C (resp. H) satisfies (11') & (2) (resp. (11') & (1'3)) but not (1).

In conclusion, all the subvarieties of $\boldsymbol{P}_{6,0}$ can be characterised by axioms belonging to $M \cup \{(3)\}$.

Step 4. The subvarieties of $\boldsymbol{P}_{6,0}$ of the form $\boldsymbol{P}_{m,n}$ are ordered as follows:

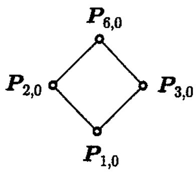


FIGURE 3

Using the results recalled in the Introduction, we obtain

$$|Si(\boldsymbol{P}_{6,0})| = D(6) + \sum_{p\,|\,3} \Psi(p) = 4 + 4 = 8 .$$

Moreover, we have $\alpha_3 = 4$ and $\alpha_1 = 2$: there are 4 simple algebras that belong properly to $\boldsymbol{P}_{6,0}$ and 2 simple algebras that belong properly to $\boldsymbol{P}_{2,0}$. Each of the subvarieties $\boldsymbol{P}_{1,0}$ and $\boldsymbol{P}_{3,0}$ contains a unique simple algebra. The description of the eight simple algebras of $\boldsymbol{P}_{6,0}$ is given in Table 2.

<table>
<tr><th>(α)</th><th>(β)</th><th colspan="2">(γ)</th><th>(δ)</th><th>(ε)</th></tr>
<tr><td>$\boldsymbol{P}_{1,0}$</td><td>A</td><td>0</td><td></td><td><table><tr><td></td><td>[1]</td><td>[1′]</td><td>[3]</td><td>[3′]</td><td>[2]</td></tr><tr><td>0</td><td>×</td><td>×</td><td>×</td><td>×</td><td>×</td></tr></table></td><td>(1)</td></tr>
<tr><td rowspan="2">$\boldsymbol{P}_{2,0}$</td><td>B</td><td>0 1</td><td rowspan="2"><table><tr><td></td><td>0</td><td>1</td></tr><tr><td>g</td><td>1</td><td>0</td></tr><tr><td>g^2</td><td>0</td><td>1</td></tr><tr><td>g^3</td><td>1</td><td>0</td></tr></table></td><td><table><tr><td></td><td>[1]</td><td>[1′]</td><td>[3]</td><td>[3′]</td><td>[2]</td></tr><tr><td>0</td><td></td><td></td><td></td><td></td><td>×</td></tr><tr><td>1</td><td></td><td></td><td></td><td></td><td>×</td></tr></table></td><td>(2)</td></tr>
<tr><td>C</td><td>1
0</td><td><table><tr><td></td><td>[1]</td><td>[1′]</td><td>[3]</td><td>[3′]</td><td>[2]</td></tr><tr><td>0</td><td>×</td><td></td><td>×</td><td></td><td>×</td></tr><tr><td>1</td><td></td><td>×</td><td></td><td>×</td><td>×</td></tr></table></td><td>(11′, 2)</td></tr>
<tr><td>$\boldsymbol{P}_{3,0}$</td><td>D</td><td>0 1 2</td><td><table><tr><td></td><td>0</td><td>1</td></tr><tr><td>g</td><td>1</td><td>2</td></tr><tr><td>g^2</td><td>2</td><td>0</td></tr><tr><td>g^3</td><td>0</td><td>1</td></tr></table></td><td><table><tr><td></td><td>[1]</td><td>[1′]</td><td>[3]</td><td>[3′]</td><td>[2]</td></tr><tr><td>0</td><td></td><td></td><td>×</td><td>×</td><td></td></tr><tr><td>1</td><td></td><td></td><td>×</td><td>×</td><td></td></tr></table></td><td>(3)</td></tr>
<tr><td rowspan="4">$\boldsymbol{P}_{6,0}$</td><td>E</td><td>0 1 2 3 4 5
$\Gamma(0) = \emptyset$</td><td rowspan="4"><table><tr><td></td><td>0</td><td>1</td></tr><tr><td>g</td><td>1</td><td>2</td></tr><tr><td>g^2</td><td>2</td><td>3</td></tr><tr><td>g^3</td><td>3</td><td>4</td></tr></table></td><td><table><tr><td></td><td>[1]</td><td>[1′]</td><td>[3]</td><td>[3′]</td><td>[2]</td></tr><tr><td>0</td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>1</td><td></td><td></td><td></td><td></td><td></td></tr></table></td><td>(–)</td></tr>
<tr><td>F</td><td>3 1 5
0 4 2
$\Gamma(0) = \{3\}$</td><td><table><tr><td></td><td>[1]</td><td>[1′]</td><td>[3]</td><td>[3′]</td><td>[2]</td></tr><tr><td>0</td><td></td><td></td><td>×</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td><td></td><td>×</td><td></td></tr></table></td><td>(33′)</td></tr>
<tr><td>G</td><td>1 3 5
0 2 4
$\Gamma(0) = \{1\}$</td><td><table><tr><td></td><td>[1]</td><td>[1′]</td><td>[3]</td><td>[3′]</td><td>[2]</td></tr><tr><td>0</td><td>×</td><td></td><td></td><td></td><td></td></tr><tr><td>1</td><td></td><td>×</td><td></td><td></td><td></td></tr></table></td><td>(11′)</td></tr>
<tr><td>H</td><td>1 3 5
0 2 4
$\Gamma(0) = \{1, 3\}$</td><td><table><tr><td></td><td>[1]</td><td>[1′]</td><td>[3]</td><td>[3′]</td><td>[2]</td></tr><tr><td>0</td><td>×</td><td></td><td>×</td><td></td><td></td></tr><tr><td>1</td><td></td><td>×</td><td></td><td>×</td><td></td></tr></table></td><td>(11′, 1′3)</td></tr>
</table>

TABLE 2

The equational bases of column (ε) are in irreducible form. Furthermore, through column (δ) one can verify the exactness of Table 1: the elements 0 and 1 of H and D show that $[1,3]$, $[1',3']$ and $[3,3']$ are sterile; $[1,1']$, $[1,3']$ and $[1',3]$ are fruitful since A is the only subdirectly irreducible in which these pairs are satisfied.

Step 5. Since we have an equational basis for each subdirectly irreducible, we can order $Si(\boldsymbol{P}_{6,0})$ as follows:

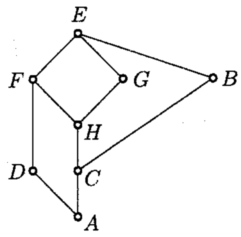


FIGURE 4

Step 6. If we include the trivial subvariety T, we have $|\Lambda(\boldsymbol{P}_{6,0})| = 20$ and the complete description of $\Lambda(\boldsymbol{P}_{6,0})$ is as follows:

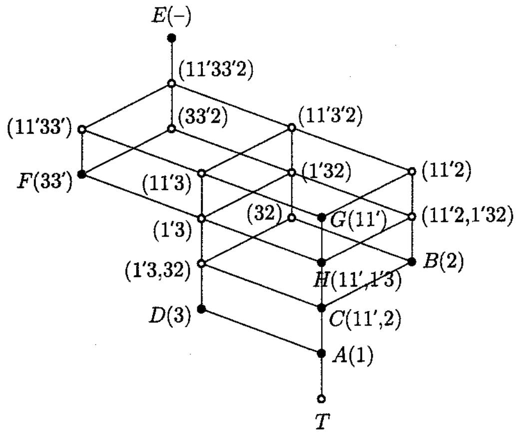


FIGURE 5

Comments. In order to compare our results with those of [3], we give the translation of our g-relations on the dual space into $\sim$-inequalities on the algebra itself. It goes without saying that every subvariety of $\boldsymbol{P}_{6,0}$ satisfies $a = \sim^6 a$.

$$\begin{array}{rl}
(1) & a \vee \sim a = 1 \,; \\
(2) & a \leq \sim^2 a \,; \\
(3) & a \vee \sim^3 a = 1 \,; \\
(11') & a \wedge \sim a \leq b \vee \sim b \,; \\
(1'3) & a \wedge \sim a \leq b \vee \sim^3 b \,; \\
(3'2) & a \wedge \sim^3 a \leq \sim^2 a \,; \\
(33') & a \wedge \sim^3 a \leq b \vee \sim^3 b \,; \\
(11'2) & a \wedge \sim a \leq \sim^2 a \vee b \vee \sim b \,; \\
(11'3) & a \wedge \sim a \wedge \sim^3 a \leq b \vee \sim b \,; \\
(1'32) & a \wedge \sim a \leq \sim^2 a \vee b \vee \sim^3 b \,; \\
(33'2) & a \wedge \sim^3 a \leq \sim^2 a \vee b \vee \sim^3 b \,; \\
(11'3'2) & a \wedge \sim a \wedge \sim^3 a \leq \sim^2 a \vee b \vee \sim b \,; \\
(11'33') & a \wedge \sim a \wedge \sim^3 a \leq b \vee \sim b \vee \sim^3 b \,; \\
(11'33'2) & a \wedge \sim a \wedge \sim^3 a \leq \sim^2 a \vee b \vee \sim b \vee \sim^3 b \,.
\end{array}$$

All these $\sim$-inequalities are basic and involve at most two variables. By Theorem 8 they are self-dual, a fact that clearly appears in (11′), (33′) and (11′33′) and that can be easily proved in all cases by the following process:

apply $\sim^3$ to both members of the inequality ;
change a into $\sim^3 a$ and b into $\sim^3 b$.

We show it for the first and last axioms of the list above.

Applying $\sim^3$ to both members of (1), we obtain

$$\sim^3 a \wedge \sim^4 a = 0 \,.$$

Changing a into $\sim^3 a$ yields

$$a \wedge \sim a = 0 \,,$$

that is the dual of (1).

Similarly, applying $\sim^3$ to both members of (11′33′2) we obtain

$$\sim^3 a \vee \sim^4 a \vee a \geq \sim^5 a \wedge \sim^3 b \wedge \sim^4 b \wedge b \,.$$

Changing a into $\sim^3 a$ yields

$$a \vee \sim a \vee \sim^3 a \geq \sim^2 a \wedge b \wedge \sim b \wedge \sim^3 b \,,$$

that is the dual of (11′33′2).

The preceding considerations show that the g-relations are much more tractable than the $\sim$-inequalities.

Now we compare our results with those of M. Ramalho. We first note that two axioms of [3] involve only one variable but are not simple. These are

$$(\alpha)\quad a \wedge \sim a \leq \sim^2 a \vee \sim^3 a$$

and

$$(\beta)\quad a \wedge \sim^3 a \leq \sim^2 a \vee \sim a\,.$$

We shall prove that $(\alpha) = (11'2)$.

That $(11'2)$ implies (α) is immediate: it suffices to change b into $\sim^2 a$.

That (α) implies $(11'2)$ is more arduous. Substitute $a \vee (\sim^5 a \wedge \sim^4 b)$ for a in (α):

$$\left[a \vee (\sim^5 a \wedge \sim^4 b)\right] \wedge \left[\sim a \wedge (a \vee \sim^5 b)\right] \leq \sim^2 a \vee (\sim a \wedge b) \vee \left[\sim^3 a \wedge (\sim^2 a \vee \sim b)\right].$$

The first member is greater than or equal to $a \wedge \sim a$, whereas the second is less than or equal to $\sim^2 a \vee b \vee \sim b$. Hence $(11'2)$ is satisfied.

Some of the equational bases obtained in [3] are not correct. For instance, the algebra dual to F is denoted in [3] by $C_{3,3}$ and characterised by $(11'33')$ and (β). But we can show that (β) is not satisfied. In fact, the algebra dual to F has the following Hasse diagram:

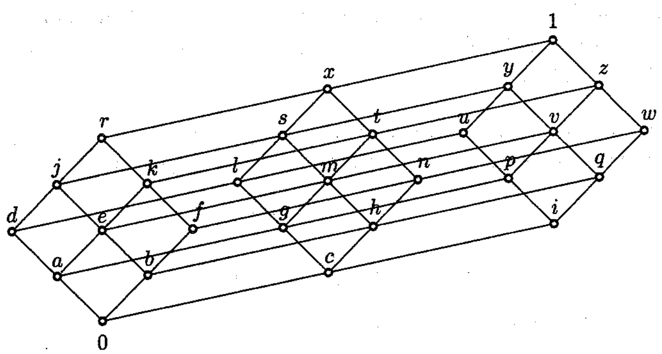


FIGURE 6

The operation $\sim$ is defined as follows:

λ	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q	r	s	t	u	v	w	x	y	z	1
$\sim\lambda$	1	y	x	z	u	s	r	v	t	w	l	j	p	m	k	q	n	d	g	e	i	h	f	a	c	b	0

The element j does not satisfy (β) since $\sim j = l$, $\sim^2 j = p$, $\sim^3 j = q$ and $b = j \wedge q \not\leq p \vee l = u$.

The algebra dual to G has a diagram isomorphic to the free distributive lattice on 3 generators:

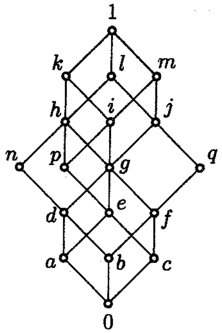


FIGURE 7

The operation $\sim$ is defined as follows:

λ	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q	1
$\sim\lambda$	1	l	m	k	j	h	i	g	f	d	e	b	c	a	q	n	p	0

For this algebra, M. Ramalho gives four equational bases: (11′), (1′3), (13′) and (α). First, (1′3) and (13′) are dual, hence equivalent. But they are not satisfied: $g \wedge \sim g \not\leq p \vee \sim^3 p$ since $g \not\leq p$. Second, $(\alpha) = (11'2)$ is the equational basis of $B \vee G$. Third, $(11') \Rightarrow (\alpha)$ but not conversely. In conclusion, the only equational basis of the algebra dual to G is (11′).

We can also say that some equational bases of [3] are redundant. For instance, the subvariety $B \vee D$ is denoted by $\mathcal{P}_{3,0} \vee \mathcal{V}(C_3)$ and characterised by

$$a \wedge \sim a \wedge \sim^3 a \leq \sim^2 a \vee b \vee \sim b \,,$$
$$a \wedge \sim^3 a \leq b \vee \sim b \,,$$
$$a \wedge \sim^3 a \leq \sim^2 a \vee b \vee \sim b \,.$$

Obviously, the second $\sim$-inequality implies the others.

4. THE SUBVARIETY $\boldsymbol{P}_{10,0}$

Step 1. There are in $\boldsymbol{P}_{10,0}$ seven non-equivalent g-relations which form an antichain when they are ordered by implication:

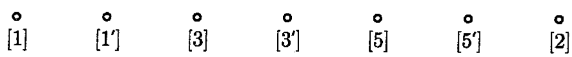


FIGURE 1′

Step 2.

&	[1]	[1′]	[3]	[3′]	[5]	[5′]	[2]
[1]	⋱	A	/	A	/	A	[3,5]
[1′]		⋱	A	/	A	/	[3′,5′]
[3]			⋱	A	/	A	[1,5]
[3′]				⋱	A	/	[1′,5′]
[5]					⋱	/	[1,3]
[5′]						⋱	[1′,3′]
[2]							⋱

TABLE 1′

Step 3. In $\boldsymbol{P}_{10,0}$ there are 42 non-equivalent axioms; 21 contain the digit 2.

$(1) \overset{d}{=} (1') = (3) \overset{d}{=} (3') = (13) \overset{d}{=} (1'3')$;

$(5) \overset{d}{=} (5') = (15) \overset{d}{=} (1'5') = (35) \overset{d}{=} (3'5') = (135) \overset{d}{=} (1'3'5')$.

$(11') = (11'3) \overset{d}{=} (11'3')$;

$(13') \overset{d}{=} (1'3)$;

$(15') \overset{d}{=} (1'5)$;

$(33') = (133') \overset{d}{=} (1'33')$;

$(35') \overset{d}{=} (3'5)$;

$(55') = (155') \overset{d}{=} (1'55') = (355') \overset{d}{=} (3'55') = (1355') \overset{d}{=} (1'3'55')$.

$(11'5) \overset{d}{=} (11'5') = (11'35) \overset{d}{=} (11'3'5')$;

$(135') \overset{d}{=} (1'3'5)$;

$(13'5) \overset{d}{=} (1'35')$;

$(13'5') \overset{d}{=} (1'35)$;

$(33'5) \overset{d}{=} (33'5') = (133'5) \overset{d}{=} (1'33'5')$.

$(2) = (12) \overset{d}{=} (1'2) = (32) \overset{d}{=} (3'2) = (132) \overset{d}{=} (1'3'2)$;

$(52) \overset{d}{=} (5'2) = (152) \overset{d}{=} (1'5'2) = (352) \overset{d}{=} (3'5'2) = (1352) \overset{d}{=} (1'3'5'2)$.

$(11'2) = (11'32) \overset{d}{=} (11'3'2)$;

$(13'2) \overset{d}{=} (1'32)$;

$(15'2) \overset{d}{=} (1'52)$;

$(33'2) = (133'2) \overset{d}{=} (1'33'2)$;

$(35'2) \overset{d}{=} (3'52)$;

$(55'2) = (155'2) \overset{d}{=} (1'55'2) = (355'2) \overset{d}{=} (3'55'2) = (1355'2) \overset{d}{=} (1'3'55'2)$.

$(11'52) \overset{d}{=} (11'5'2) = (11'352) \overset{d}{=} (11'3'5'2)$;

$(135'2) \overset{d}{=} (1'3'52)$;

$(13'52) \overset{d}{=} (1'35'2)$;

$(13'5'2) \overset{d}{=} (1'352)$;

$(33'52) \overset{d}{=} (33'5'2) = (133'52) \overset{d}{=} (1'33'5'2)$.

$(11'33')$;

$(11'35') \stackrel{d}{=} (11'3'5)$;

$(11'55') = (11'355') \stackrel{d}{=} (11'3'55')$;

$(133'5') \stackrel{d}{=} (1'33'5)$;

$(13'55') \stackrel{d}{=} (1'355')$;

$(33'55') = (133'55') \stackrel{d}{=} (1'33'55')$.

$(11'33'5) \stackrel{d}{=} (11'33'5')$.

$(11'33'55')$.

$(11'33'2)$;

$(11'35'2) \stackrel{d}{=} (11'3'52)$;

$(11'55'2) = (11'355'2) \stackrel{d}{=} (11'3'55'2)$;

$(133'5'2) \stackrel{d}{=} (1'33'52)$;

$(13'55'2) \stackrel{d}{=} (1'355'2)$;

$(33'55'2) = (133'55'2) \stackrel{d}{=} (1'33'55'2)$.

$(11'33'52) \stackrel{d}{=} (11'33'5'2)$.

$(11'33'55'2)$.

Most equivalences are consequences of Table 1′. For instance, the equivalence $(1)=(3)$ follows from $(13) \stackrel{d}{=} (1'3') = (13)$ & $(1'3') = (1) = (3)$ since $[1,1']$, $[1,3']$, $[1',3]$, $[3,3']$ are all fruitful.

Similarly, $(5) = (135)$ is justified as follows: $(135) \stackrel{d}{=} (1'3'5') = (135)$ & $(1'3'5') \Rightarrow (5)$ since $[1,1']$, $[1,3']$, $[1,5']$, $[3,3']$, $[3,5']$ are fruitful. The implication $(5) \Rightarrow (135)$ being trivial, the proof is complete.

Figure 2′ gives the ordering of the axioms by implication. To gain in clarity we have suppressed the parentheses that embrace the axioms.

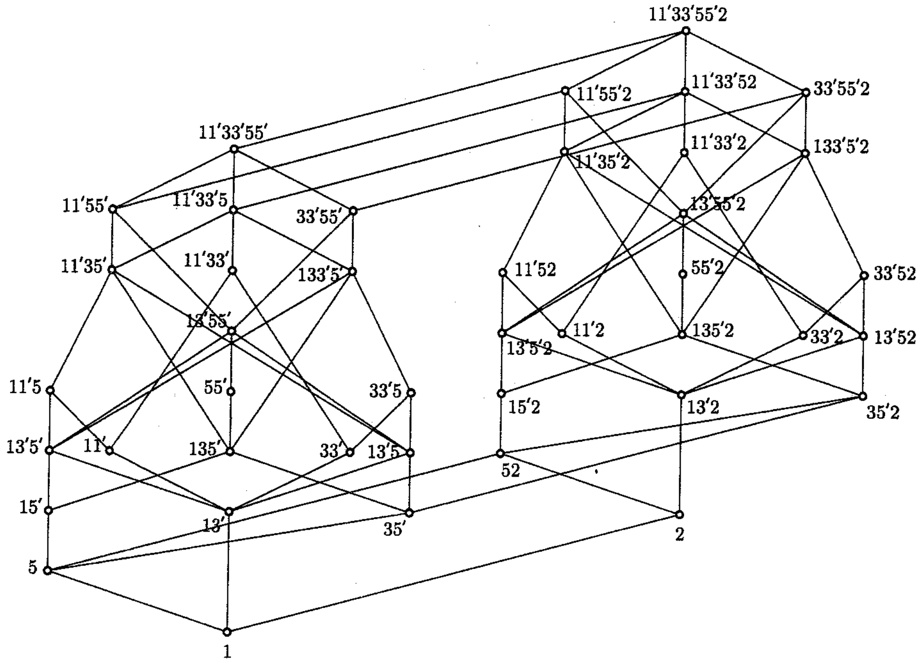


FIGURE 2′

In the poset of Figure 2′ nine elements are meet-irreducible; they form the subset $M = \{(11'33'55'2), (11'55'2), (11'33'52), (33'55'2), (11'33'2), (11'52), (55'2), (33'52), (11'33'55')\}$.

All the other axioms, except (2), (5) and (52), can be expressed as conjunctions of elements of M:

$$\begin{cases} (11'35'2) = (11'55'2) \;\&\; (11'33'52) \; ; \\ (133'5'2) = (33'55'2) \;\&\; (11'33'52) \; ; \end{cases}$$

$$(13'55'2) = (11'55'2) \;\&\; (33'55'2) \; ;$$

$$\begin{cases} (13'5'2) = (33'55'2) \;\&\; (11'52) \; ; \\ (13'52) = (11'55'2) \;\&\; (33'52) \; ; \end{cases}$$

$$(135'2) = (11'35'2) \;\&\; (55'2) = (11'55'2) \;\&\; (11'33'52) \;\&\; (55'2) \; ;$$

$$\begin{cases} (15'2) = (11'52) \;\&\; (55'2) \; ; \\ (35'2) = (33'52) \;\&\; (55'2) \; ; \end{cases}$$

$$\begin{cases} (11'2) = (11'52) \;\&\; (11'33'2) \; ; \\ (33'2) = (33'52) \;\&\; (11'33'2) \; ; \end{cases}$$

$$(13'2) = (11'2) \;\&\; (33'2) = (11'52) \;\&\; (11'33'2) \;\&\; (33'52) \; ;$$

$$(11'33'5) = (11'33'52) \;\&\; (11'33'55') \; ;$$

$$\begin{cases} (11'35') = (11'35'2) \;\&\; (11'33'55') = (11'55'2) \;\&\; (11'33'52) \;\&\; (11'33'55') \; ; \\ (133'5') = (133'5'2) \;\&\; (11'33'55') = (33'55'2) \;\&\; (11'33'52) \;\&\; (11'33'55') \; ; \end{cases}$$

$$(11'33') = (11'33'2) \;\&\; (11'33'55') \; ;$$

$$\begin{cases} (11'55') = (11'55'2) \;\&\; (11'33'55') \; ; \\ (33'55') = (33'55'2) \;\&\; (11'33'55') \; ; \end{cases}$$

$$(13'55') = (13'55'2) \;\&\; (11'33'55') = (11'55'2) \;\&\; (33'55'2) \;\&\; (11'33'55') \; ;$$

$$\begin{cases} (11'5) = (11'52) \;\&\; (11'33'55') \; ; \\ (33'5) = (33'52) \;\&\; (11'33'55') \; ; \end{cases}$$

$$(13'5') = (13'5'2) \;\&\; (11'33'55') = (33'55'2) \;\&\; (11'52) \;\&\; (11'33'55') \; ;$$

$$(135') = (135'2) \;\&\; (11'33'55') = (11'35'2) \;\&\; (55'2) \;\&\; (11'33'55') \; ;$$

$$(13'5) = (13'52) \;\&\; (11'33'55') = (11'55'2) \;\&\; (33'52) \;\&\; (11'33'55') \; ;$$

$$(55') = (55'2) \;\&\; (11'33'55') \; ;$$

$$\begin{cases} (11') = (11'2) \;\&\; (11'33'55') = (11'52) \;\&\; (11'33'2) \;\&\; (11'33'55') \; ; \\ (33') = (33'2) \;\&\; (11'33'55') = (33'52) \;\&\; (11'33'2) \;\&\; (11'33'55') \; ; \end{cases}$$

$$\begin{cases} (15') = (15'2) \;\&\; (11'33'55') = (11'52) \;\&\; (55'2) \;\&\; (11'33'55') \; ; \\ (35') = (35'2) \;\&\; (11'33'55') = (33'52) \;\&\; (55'2) \;\&\; (11'33'55') \; ; \end{cases}$$

$$(13') = (13'2) \;\&\; (11'33'55') = (11'52) \;\&\; (11'33'2) \;\&\; (33'52) \;\&\; (11'33'55') \, .$$

We also have (2) & (5) = (1).

In conclusion, all the subvarieties of $\boldsymbol{P}_{10,0}$ can be characterised by axioms belonging to $M \cup \{(2),(5),(52)\}$.

Step 4. The subvarieties of $\boldsymbol{P}_{10,0}$ of the form $\boldsymbol{P}_{m,n}$ are ordered as follows:

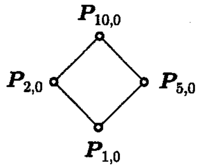


FIGURE 3′

There are 12 subdirectly irreducibles, the description of which is given in Table 2′.

Step 5. The ordering of $Si(\boldsymbol{P}_{10,0})$ is shown in Figure 4′:

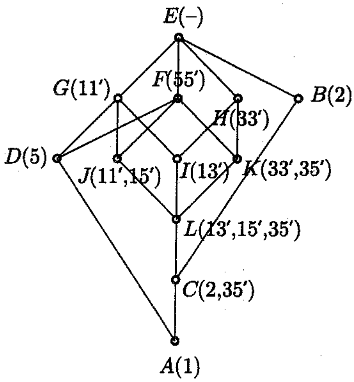


FIGURE 4′

Step 6. If we include the trivial subvariety T, we have $|\Lambda(\boldsymbol{P}_{10,0})| = 70$. The complete description of $\Lambda(\boldsymbol{P}_{10,0})$ is given in Figures 5′ and 5″.

In Figure 5′ every subvariety $X \vee Y \vee Z \vee \ldots$ is mentioned briefly as $XYZ\ldots$. On the right-hand side the name of the subvariety is not always written but can easily be found. Moreover, most lines from the left-hand side to the right-hand side have been suppressed to clarify the diagram. The subdirectly irreducibles are represented by solid circles; similarly, solid circles are used for the subvarieties $B \vee X$ where X is subdirectly irreducible.

The above remarks are also valid for Figure 5″.

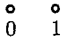
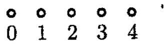
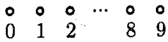
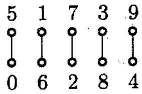
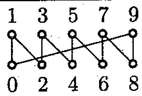
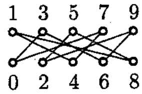
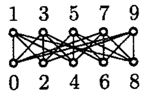
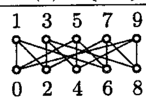
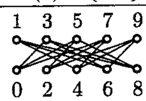
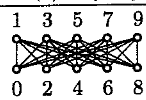
(α)	(β)	(γ)		(δ)	[1]	[1′]	[3]	[3′]	[5]	[5′]	[2]	(ε)
$\boldsymbol{P}_{1,0}$	A			0	×	×	×	×	×	×	×	(1)
$\boldsymbol{P}_{2,0}$	B			0 1 g	1 0 g^2	0 1 g^3	1 0 g^4	0 1 g^5	1 0	0							×	(2)
				1							×	
	C			0	×		×		×		×	$(2, 35')$
				1		×		×		×	×	
$\boldsymbol{P}_{5,0}$	D			0 1 g	1 2 g^2	2 3 g^3	3 4 g^4	4 0 g^5	0 1	0					×	×		(5)
				1					×	×		
$\boldsymbol{P}_{10,0}$	E	 $\Gamma(0) = \emptyset$		0 1 g	1 2 g^2	2 3 g^3	3 4 g^4	4 5 g^5	5 6	0								(−)
				1								
	F	 $\Gamma(0) = \{5\}$		0					×			$(55')$
				1						×		
	G	 $\Gamma(0) = \{1\}$		0	×							$(11')$
				1		×						
	H	 $\Gamma(0) = \{3\}$		0			×					$(33')$
				1				×				
	I	 $\Gamma(0) = \{1, 3\}$		0	×		×					$(13')$
				1		×		×				
	J	 $\Gamma(0) = \{1, 5\}$		0	×				×			$(11', 15')$
				1		×				×		
	K	 $\Gamma(0) = \{3, 5\}$		0			×		×			$(33', 35')$
				1				×		×		
	L	 $\Gamma(0) = \{1, 3, 5\}$		0	×		×		×			$(13', 15', 35')$
				1		×		×		×		

TABLE 2′

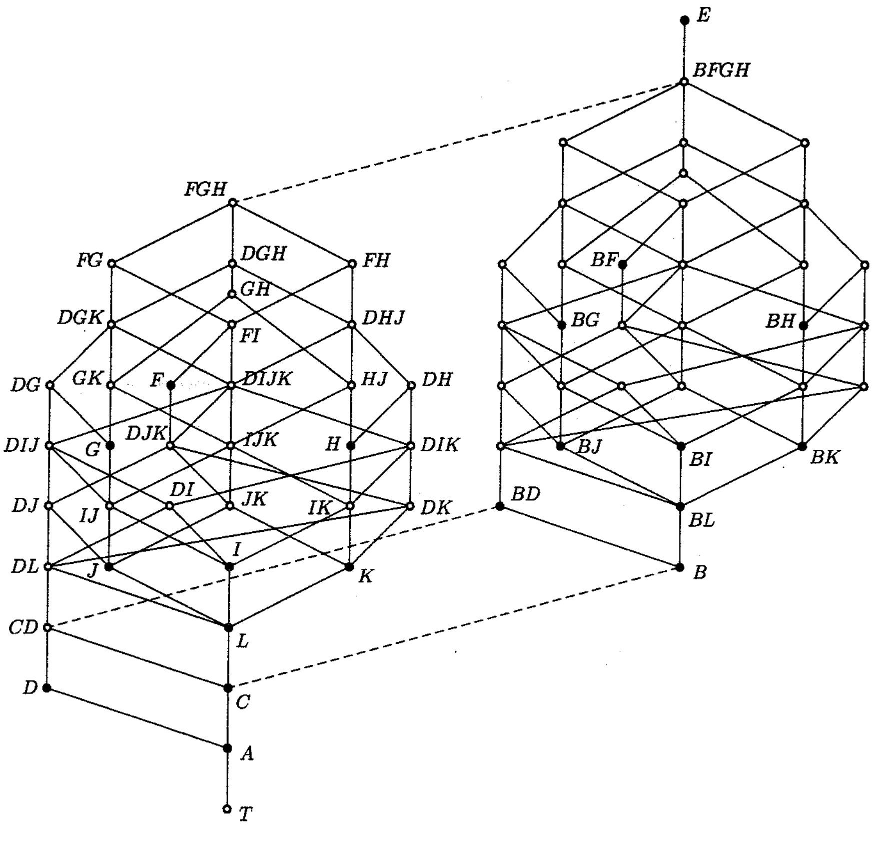

E
BFGH
FGH
FG
DGH
FH
GH
DGK
FI
DHJ
DG
GK
F
DIJK
HJ
DH
DIJ
G
DJK
IJK
H
DIK
DJ
IJ
DI
JK
IK
DK
DL
J
I
K
CD
L
D
C
A
T
BF
BG
BH
BJ
BI
BK
BD
BL
B

FIGURE 5′

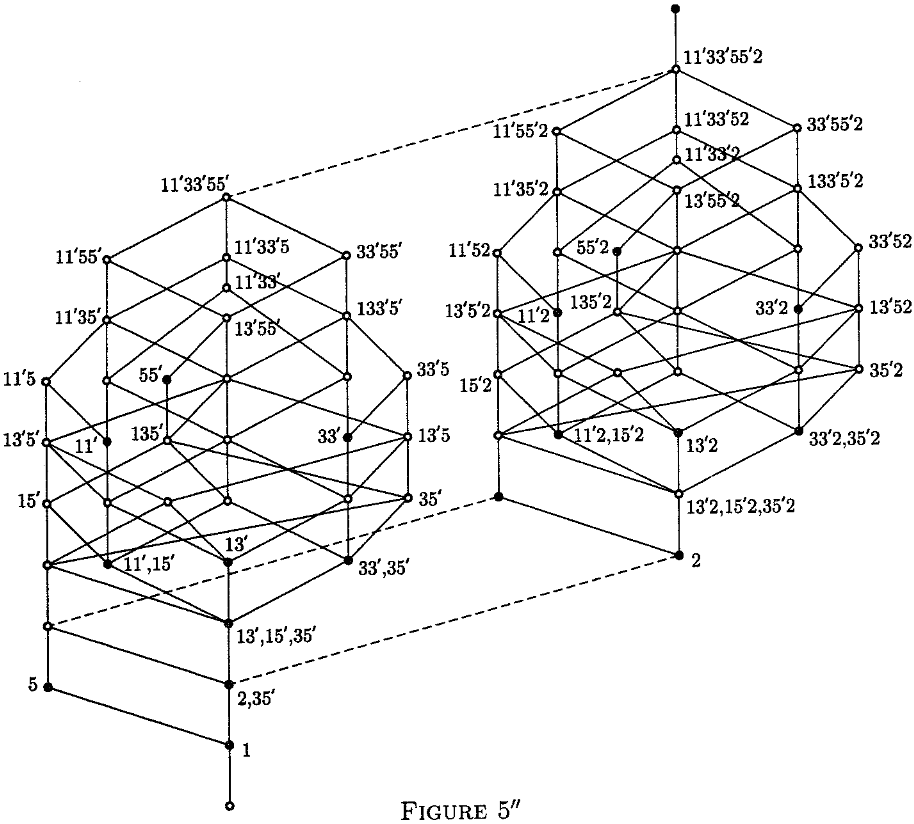


FIGURE 5″

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